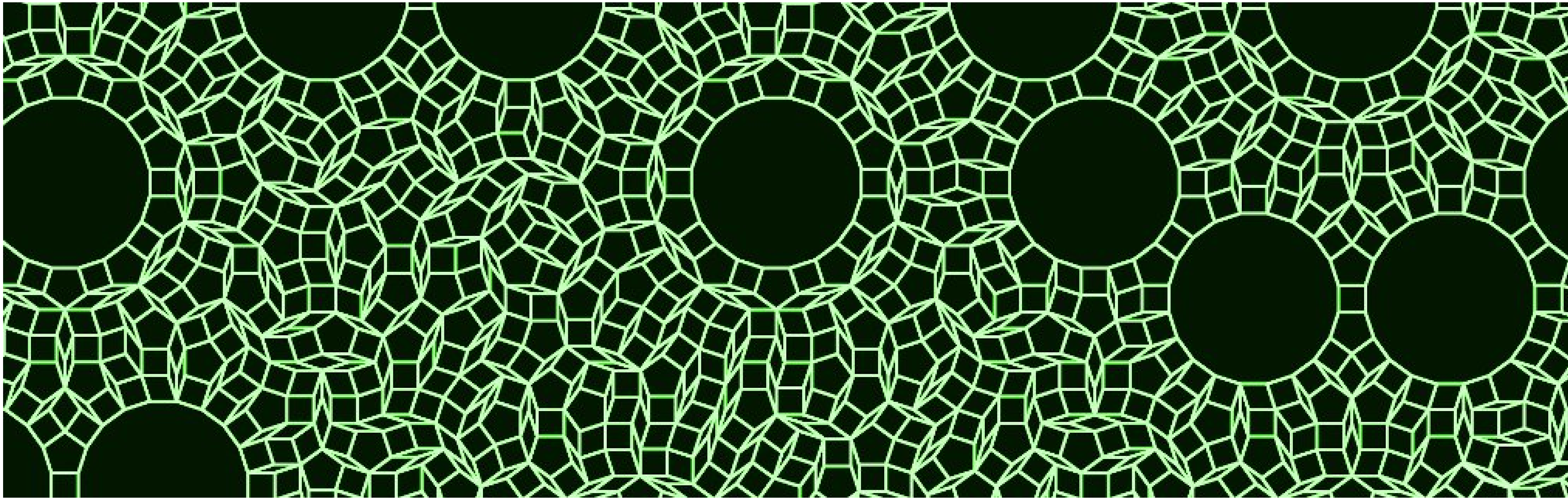
Imperfect Congruence

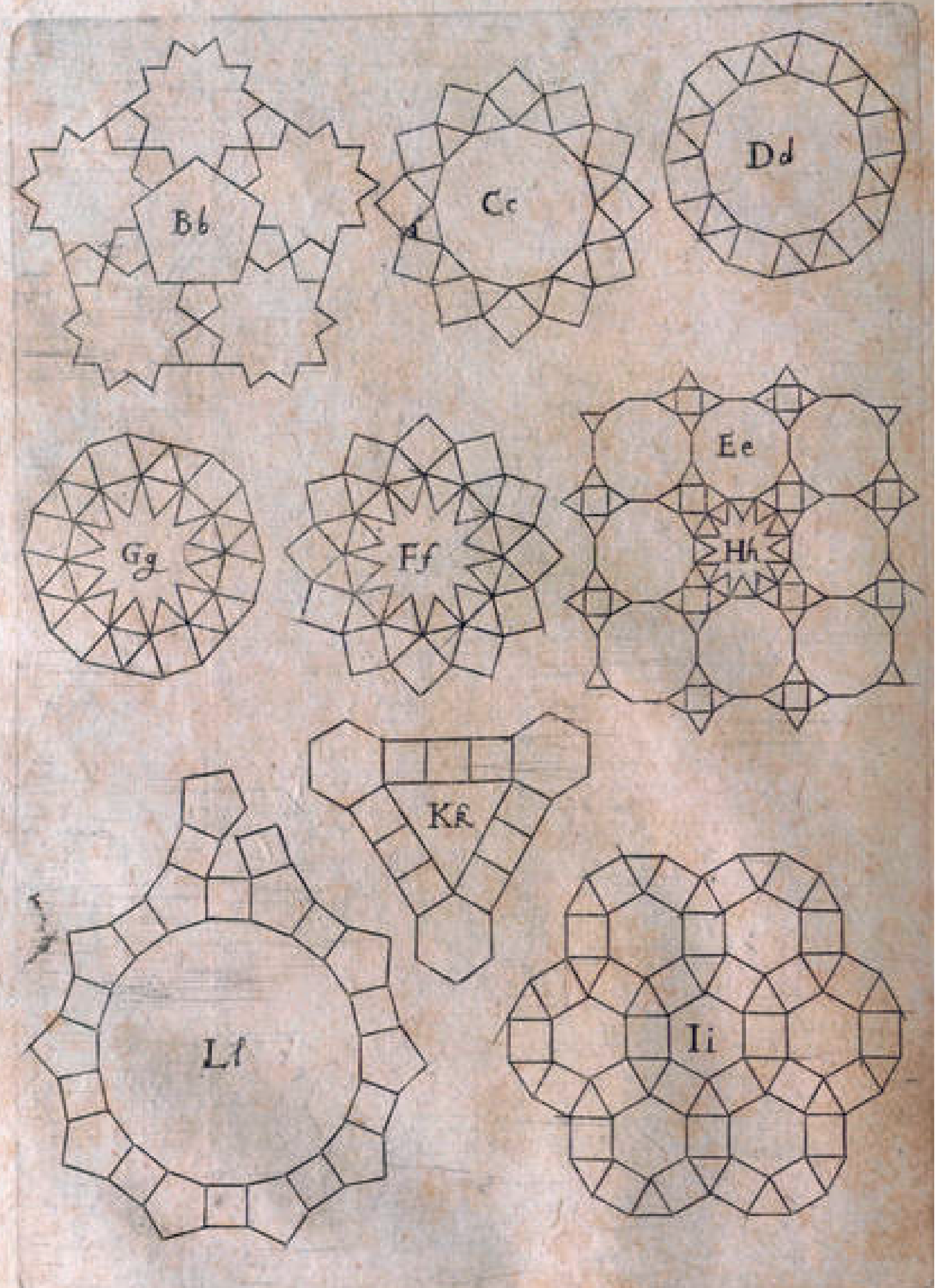


Kepler, Dürer and the mystery of the forbidden tilings

http://gruze.org/tilings

Kevin Jardine

Bridges 2013
Enchede, Netherlands
27 July 2013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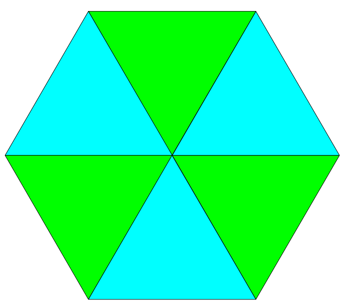
The regular congruences



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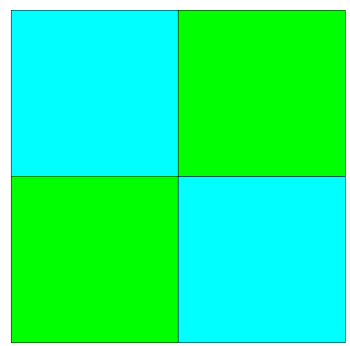

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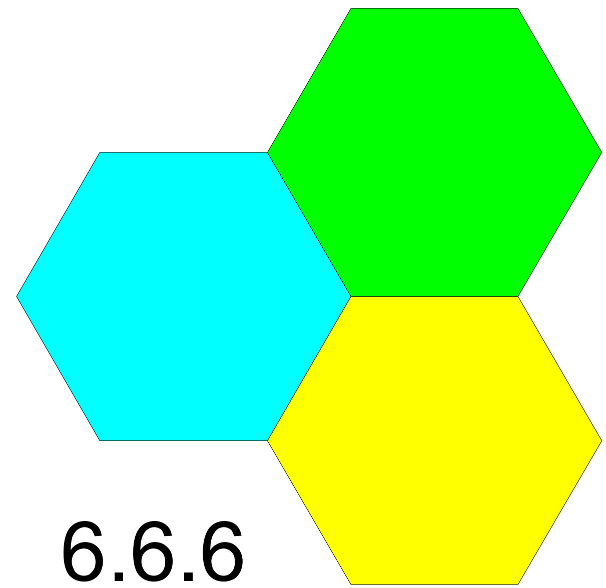
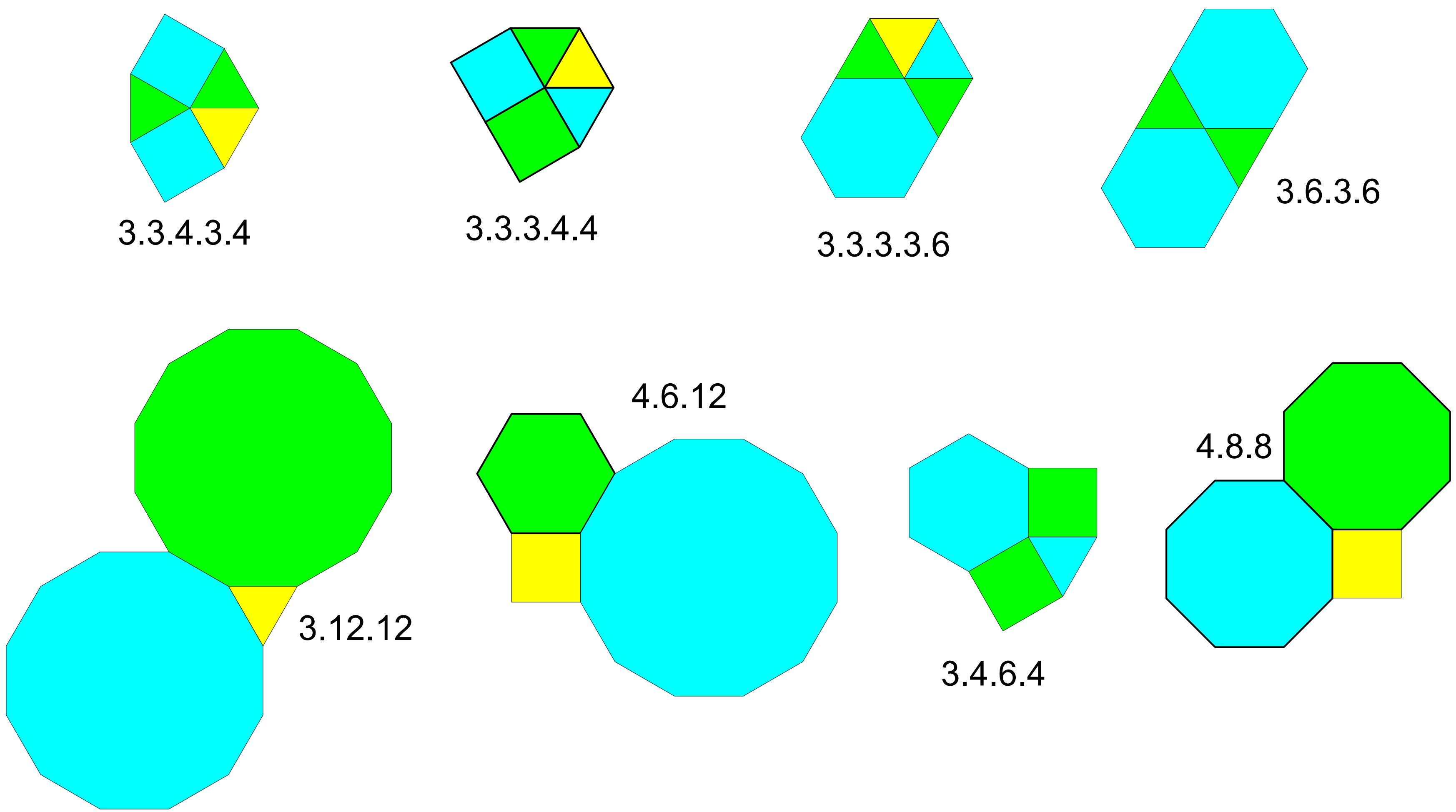
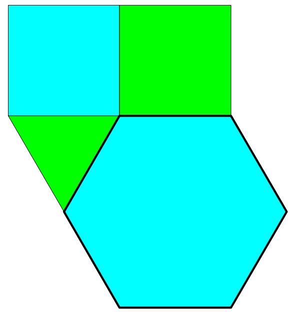


6.6.6

The semiregular congruences



The extendable imperfect congruences



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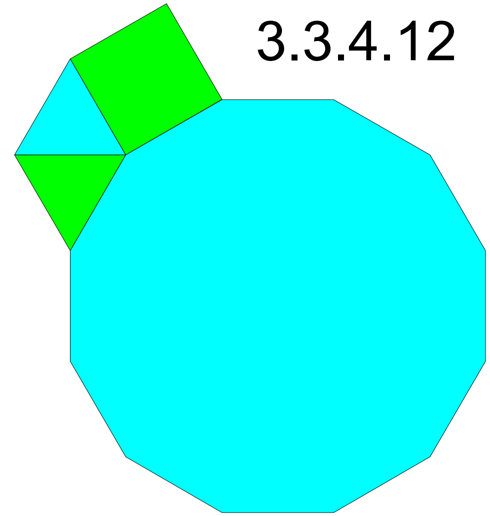
3.3.4.12



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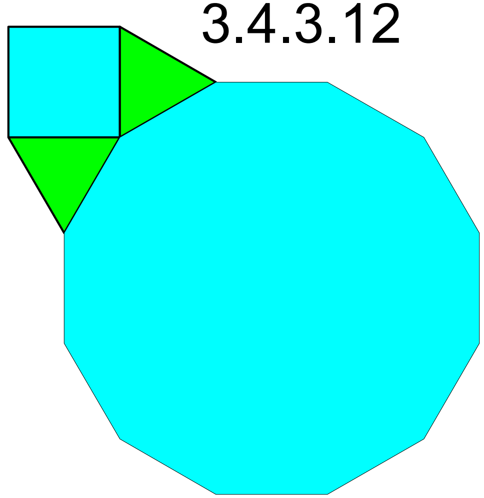

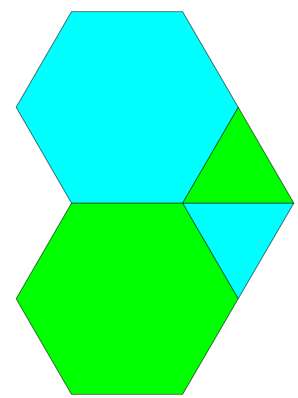

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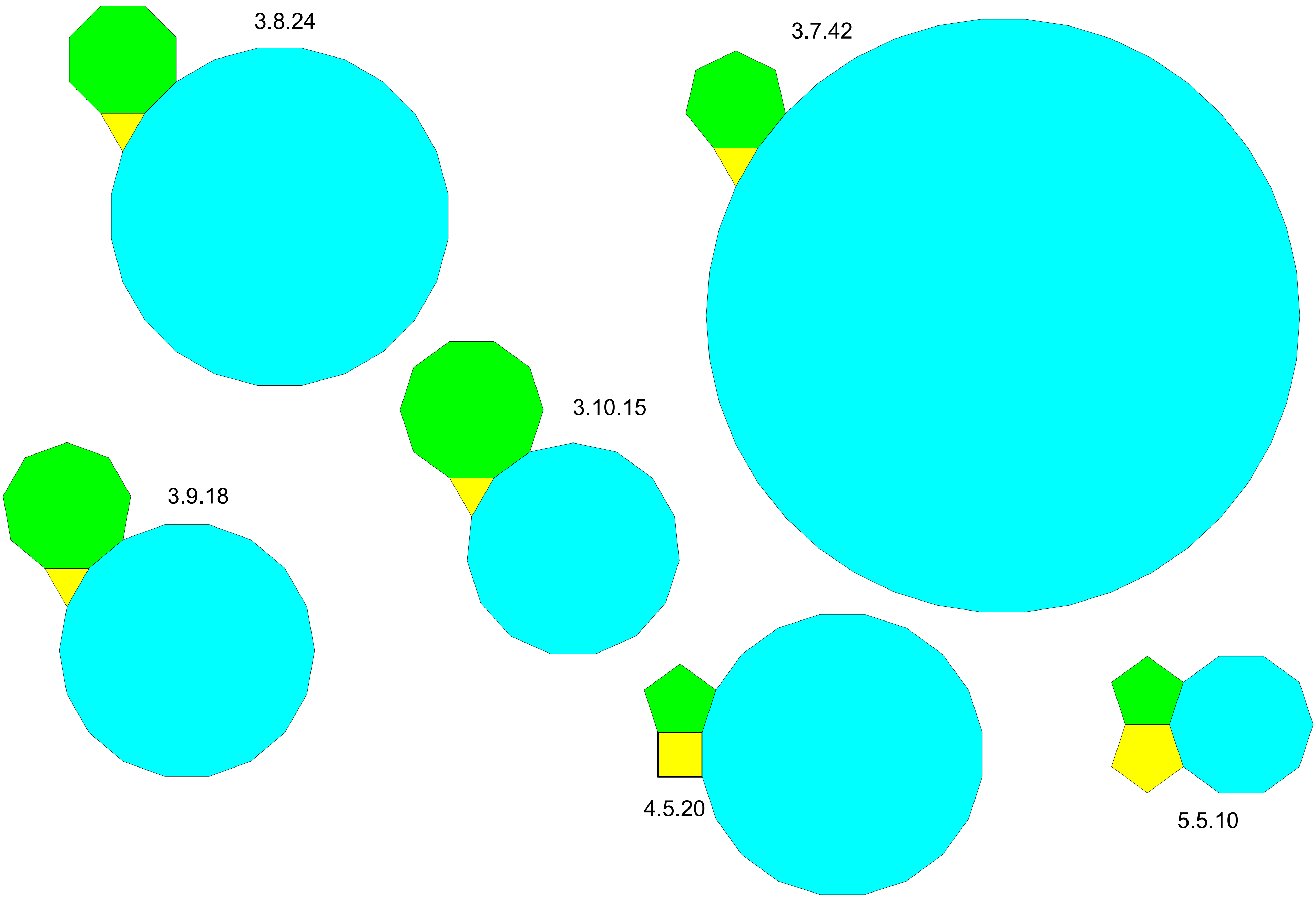
The forbidden imperfect congruences

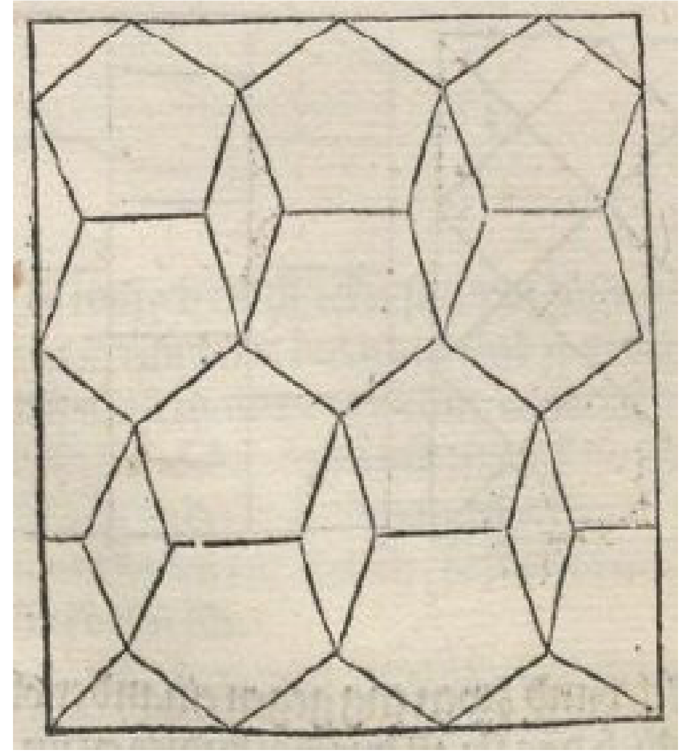


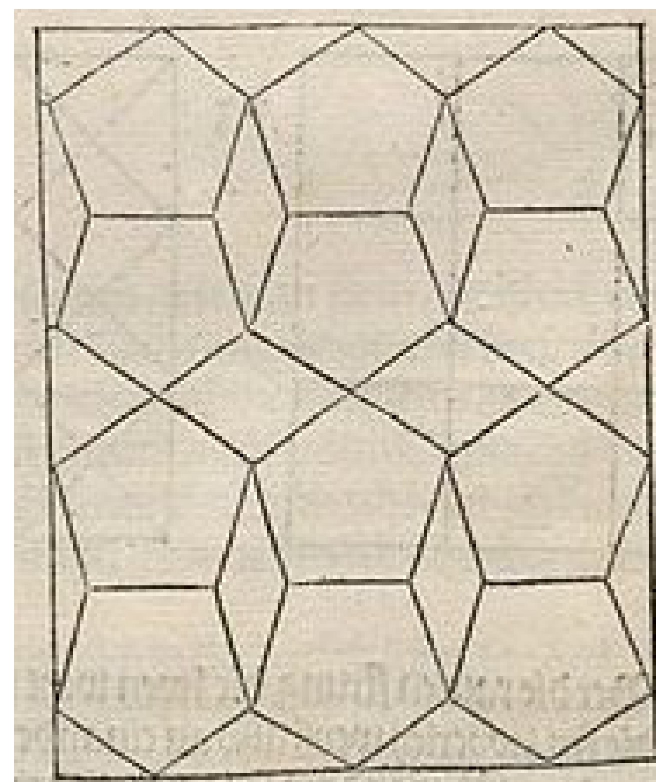
$$2x - 1 = 0$$

$$x^2 - 2 = 0$$

$$x^2 + 1 = 0$$


1500
Albertus Durerus Noricus
anno XXVIII





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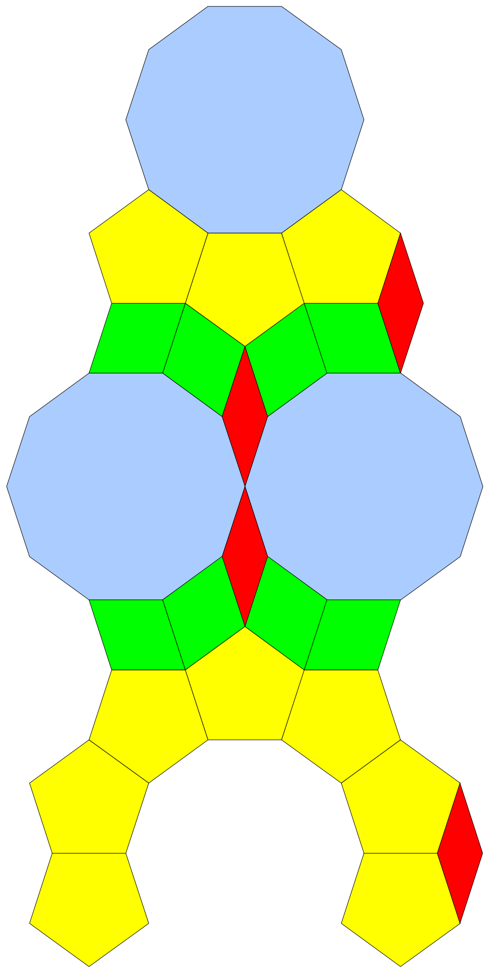

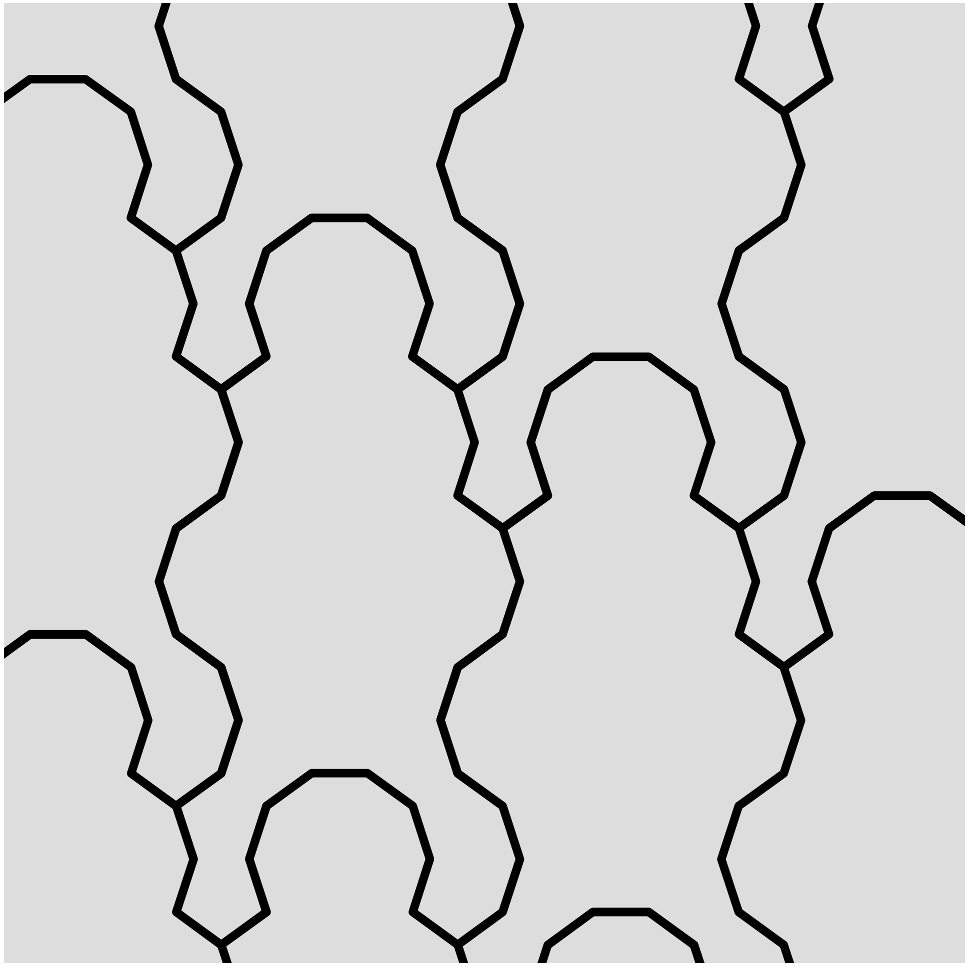

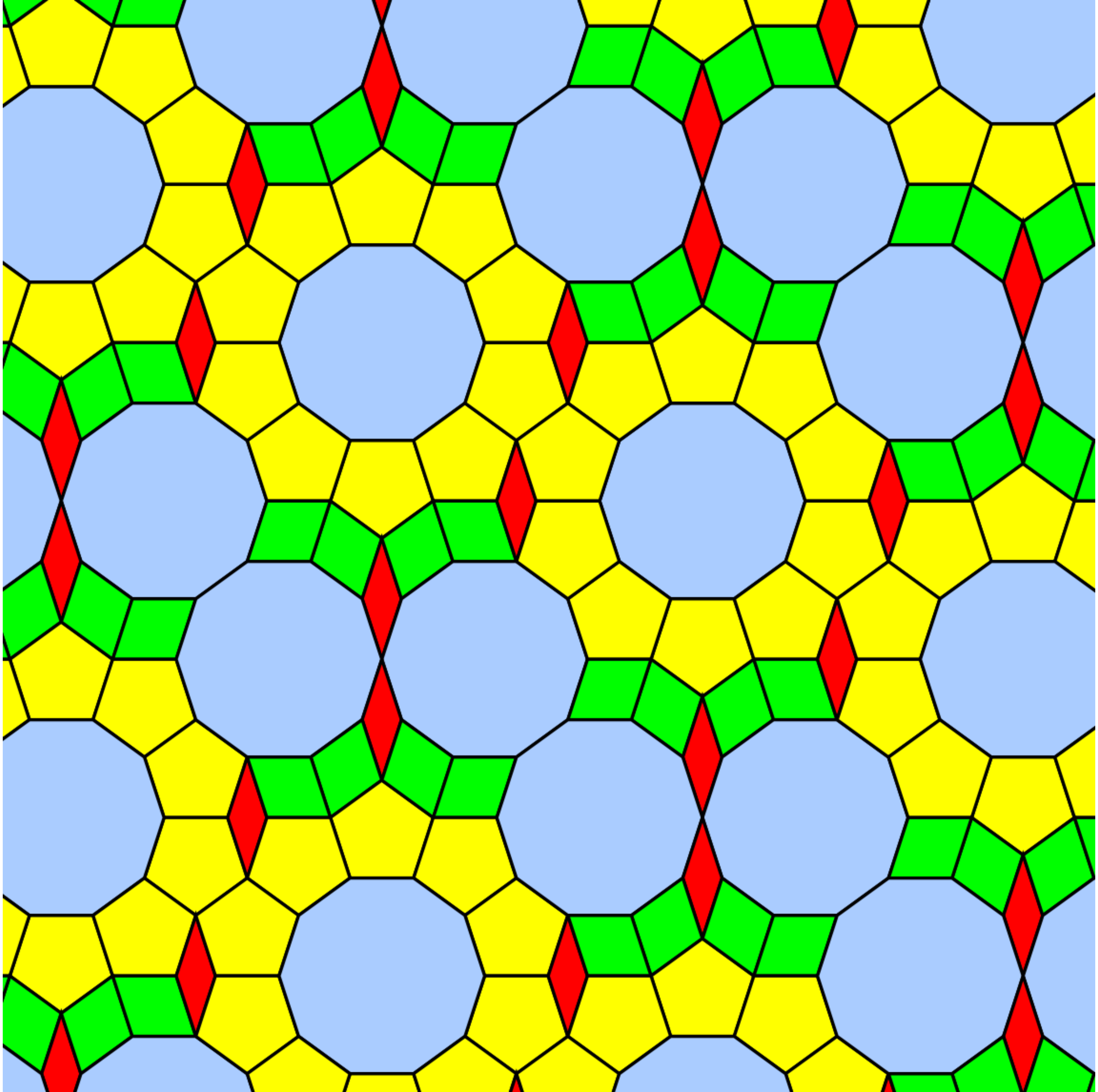

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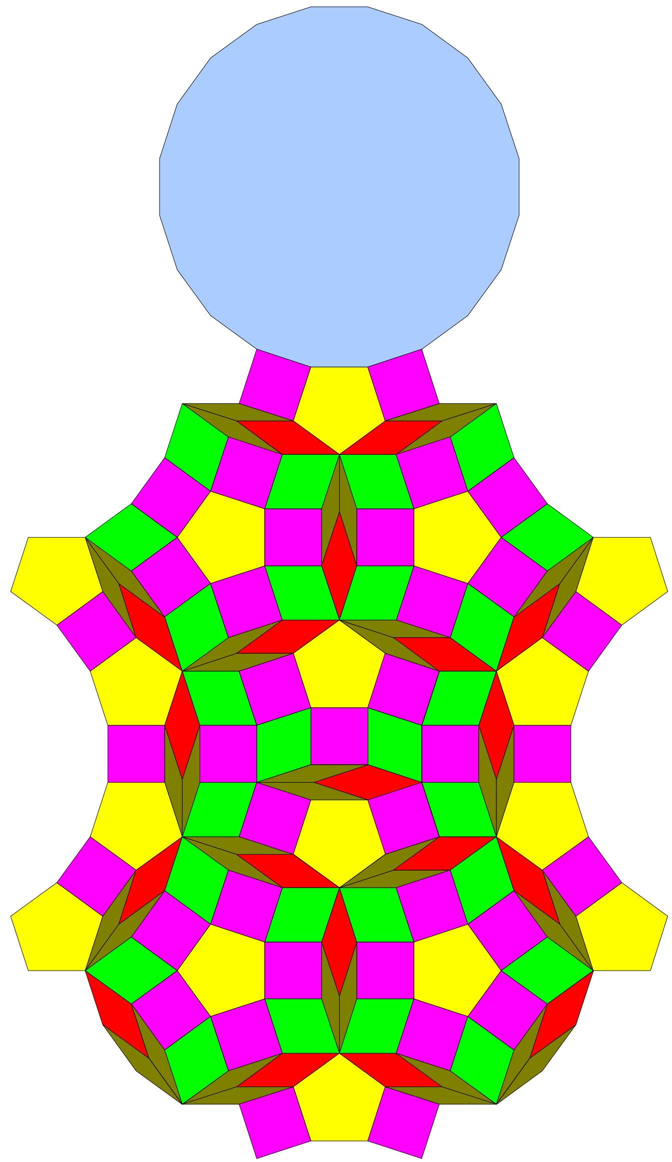

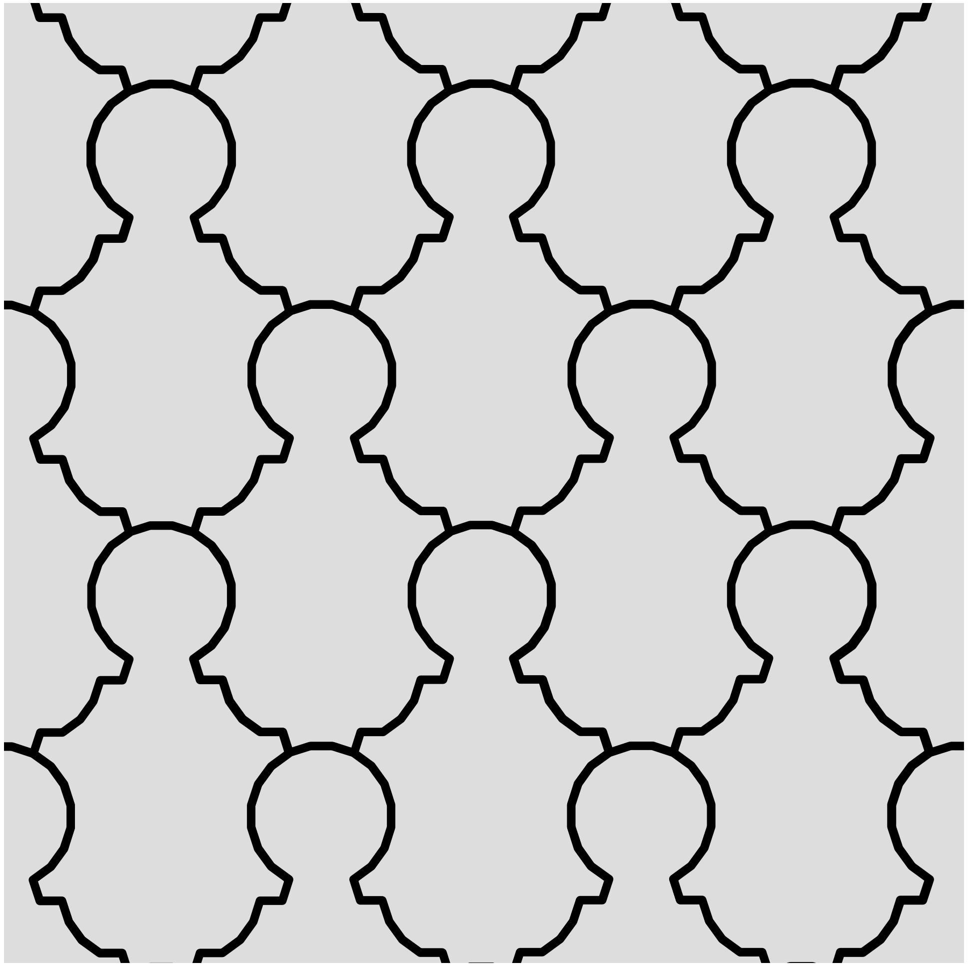

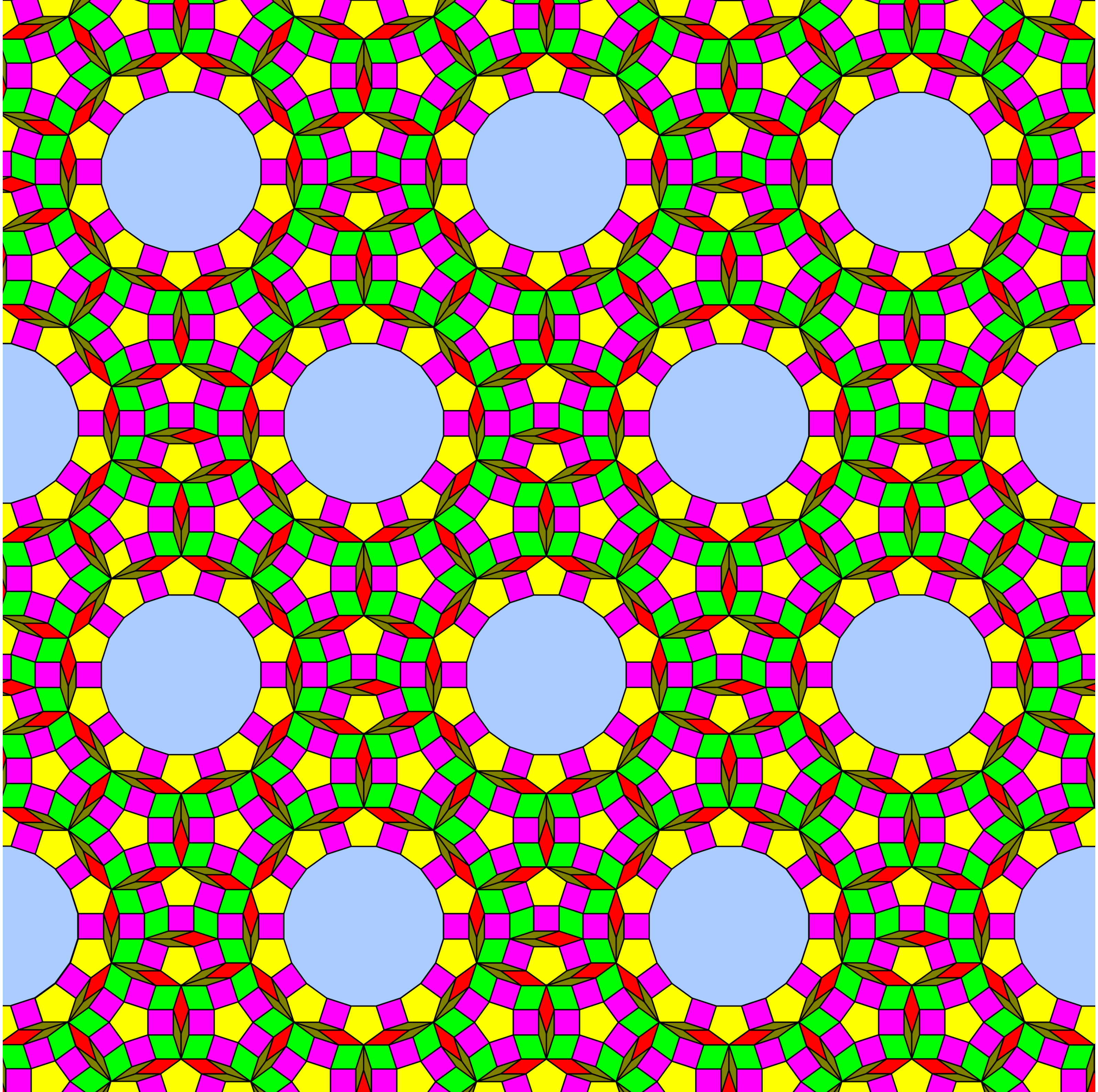

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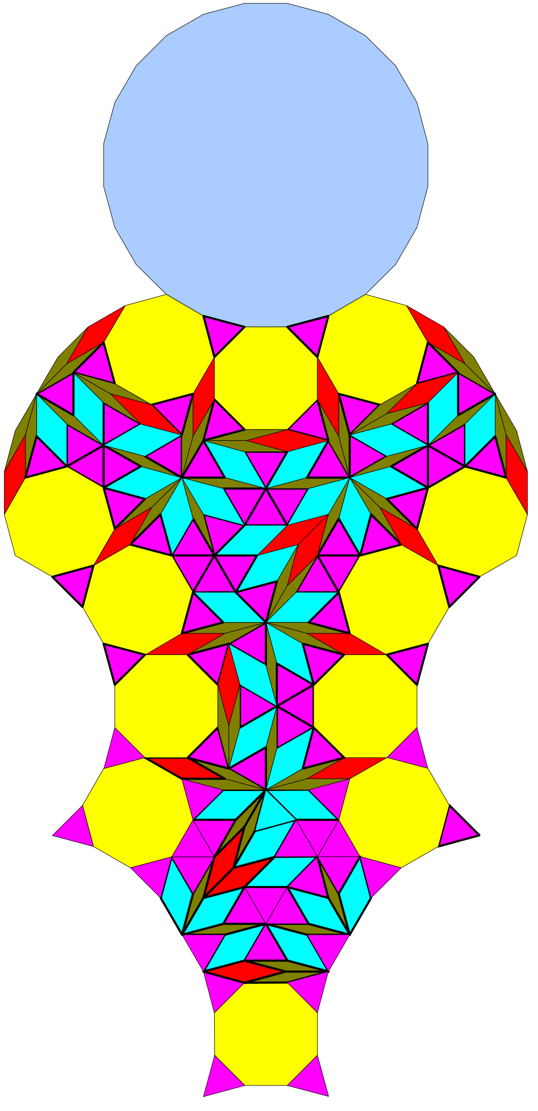

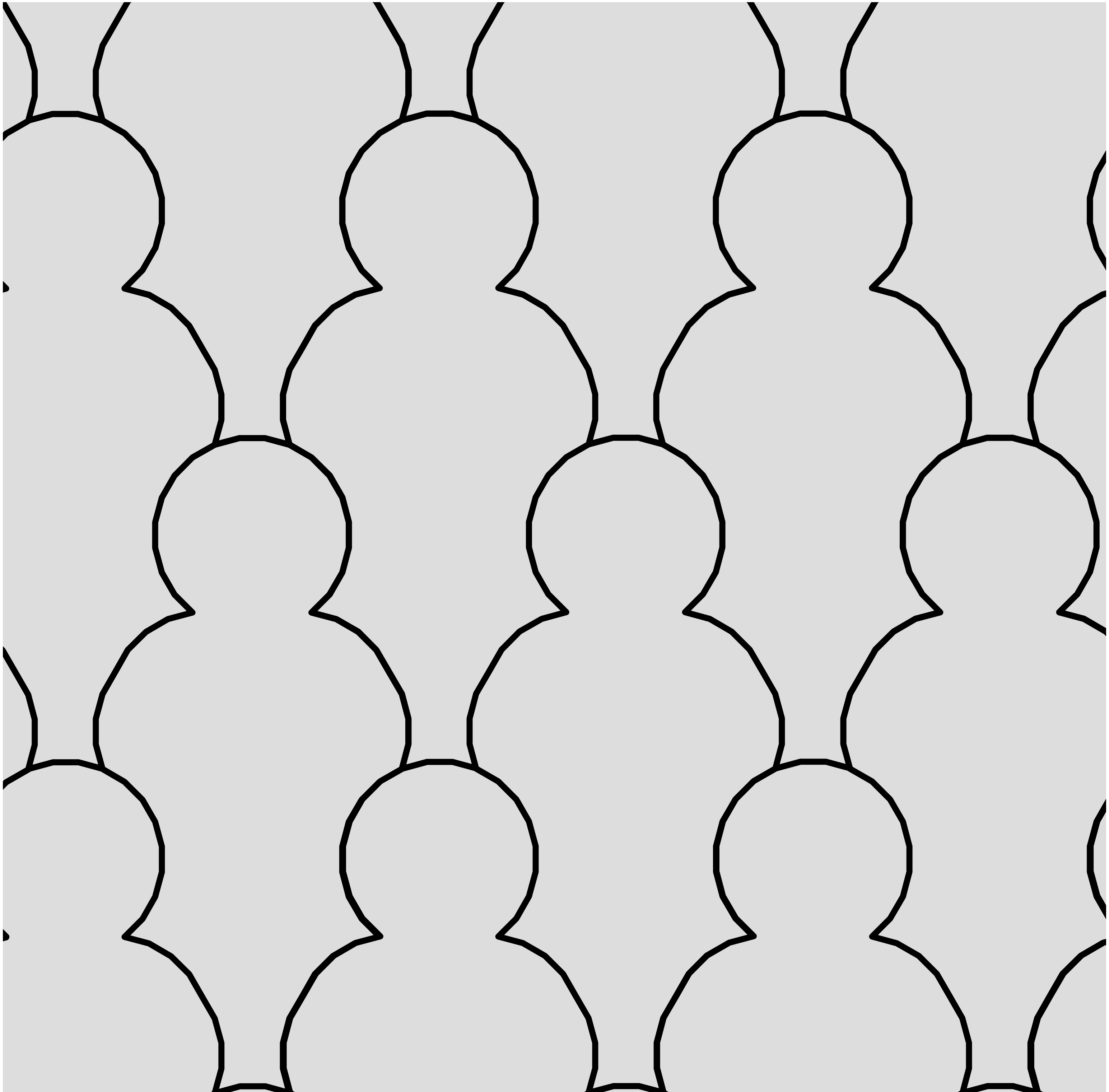

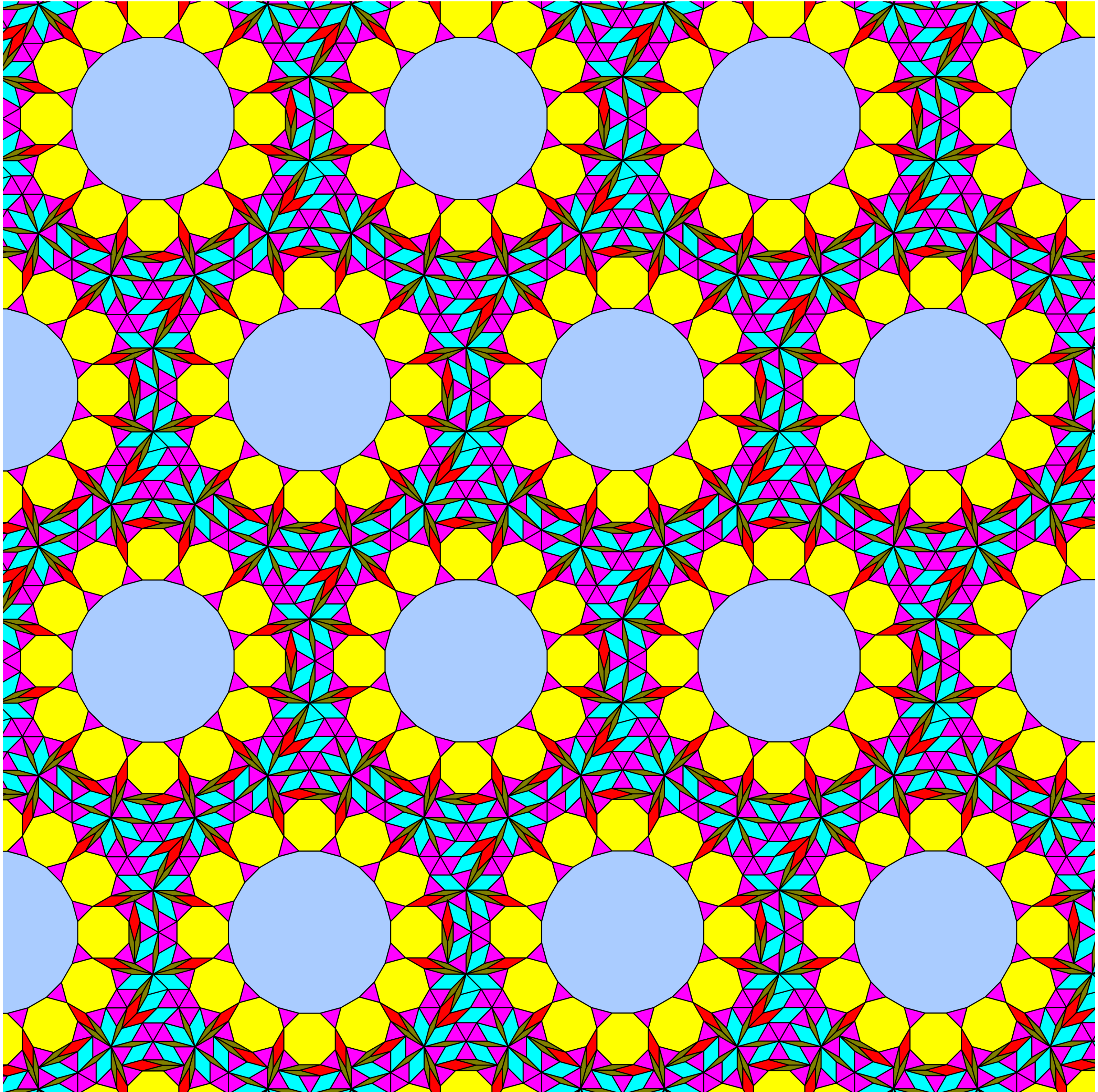

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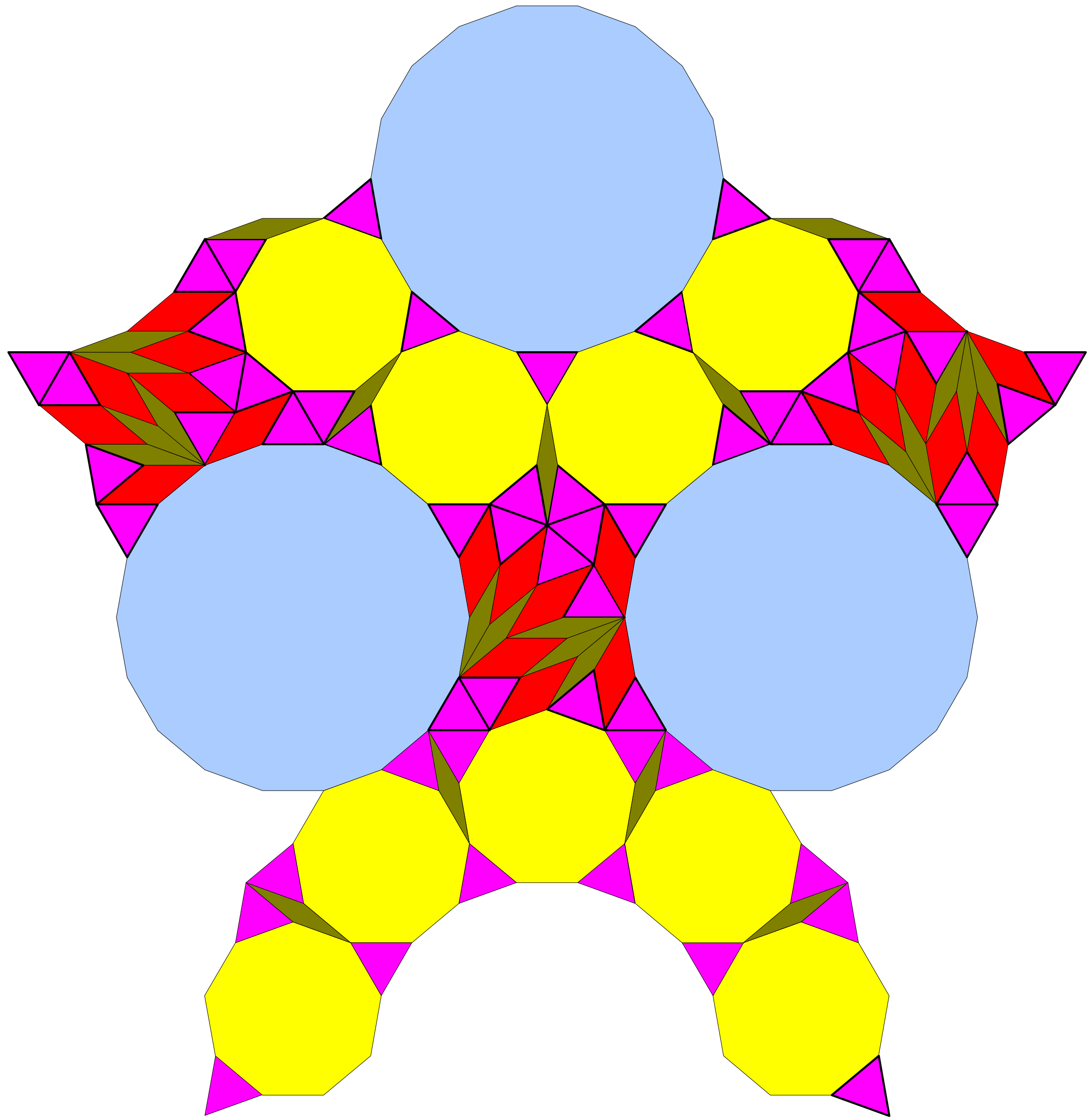

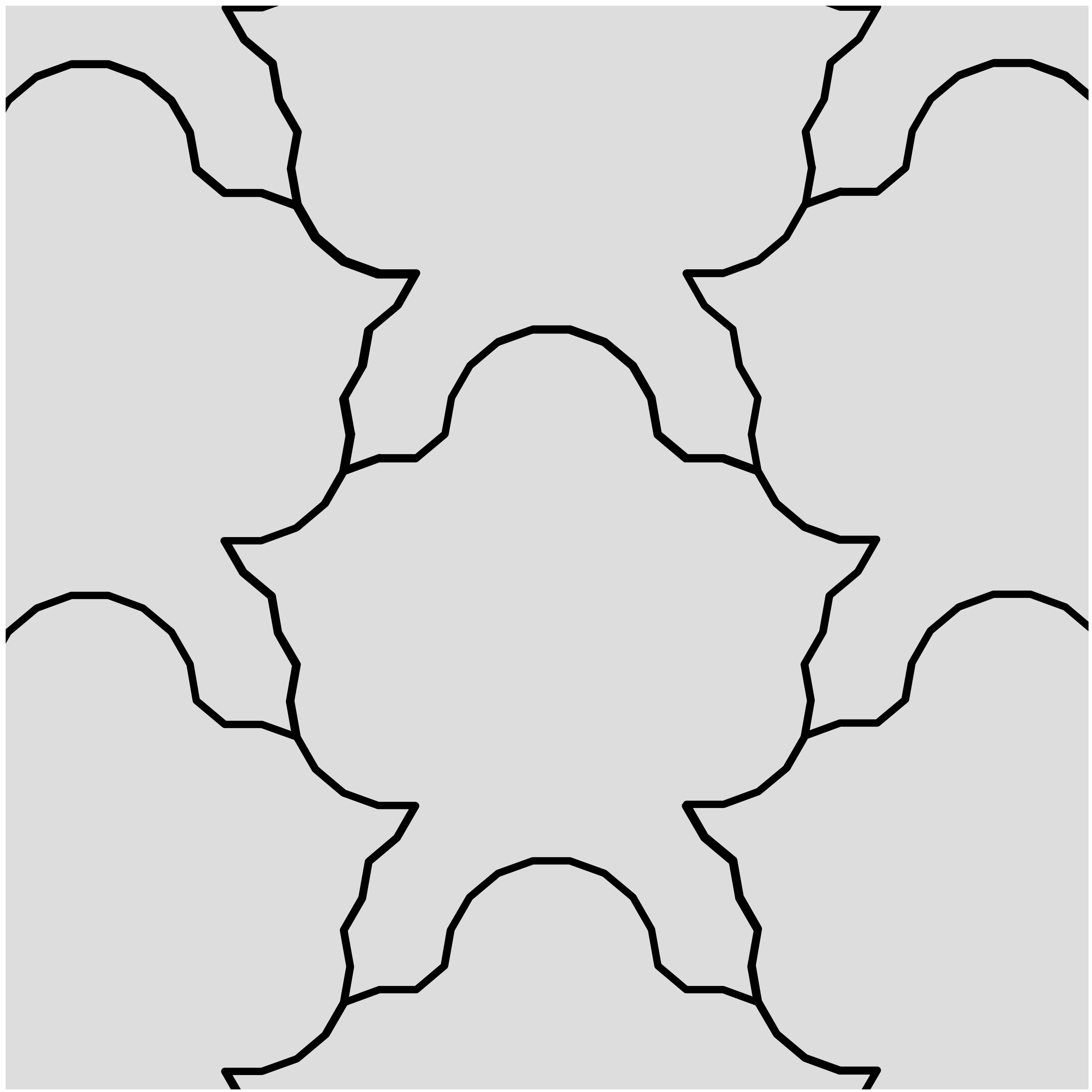

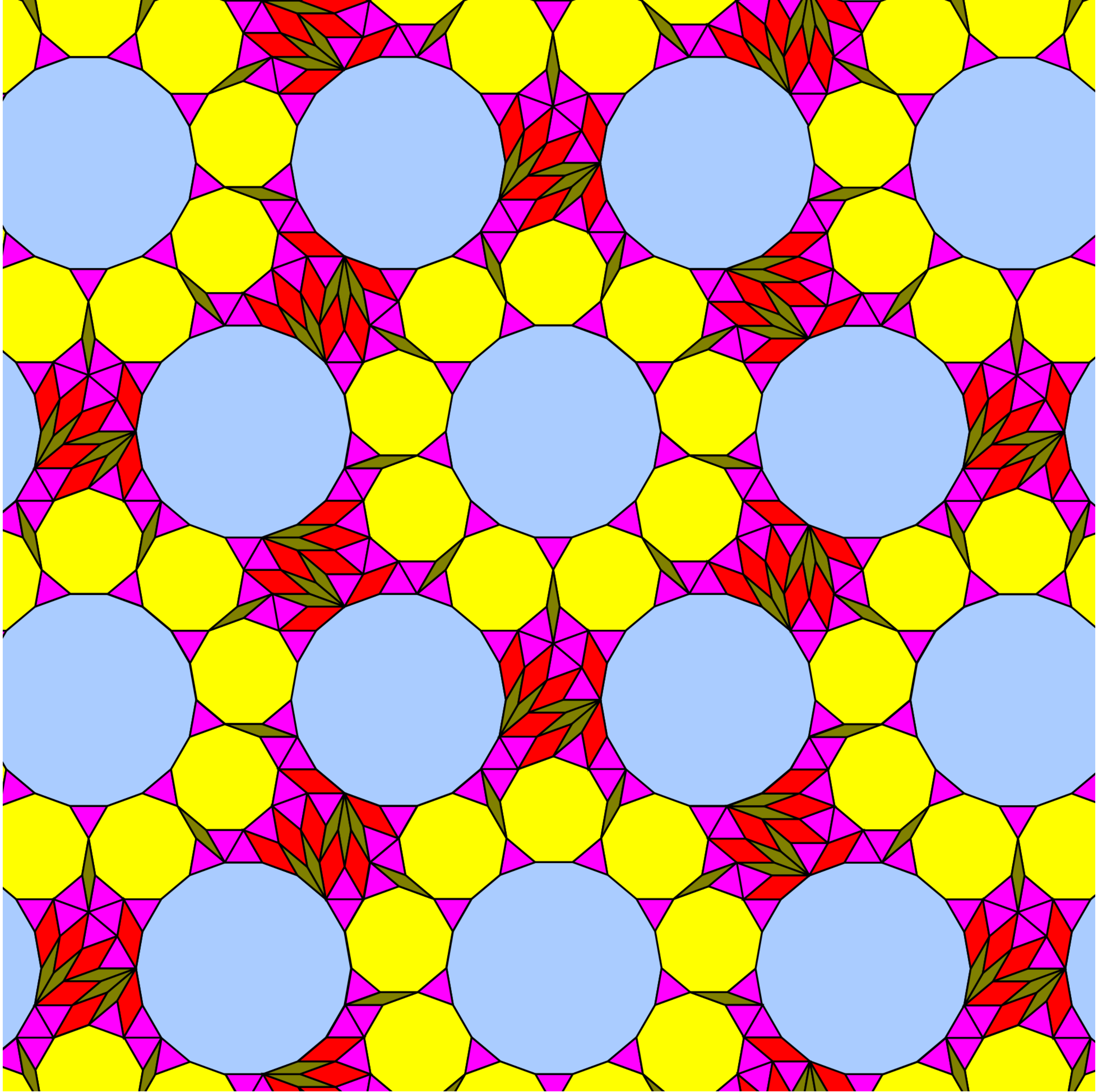

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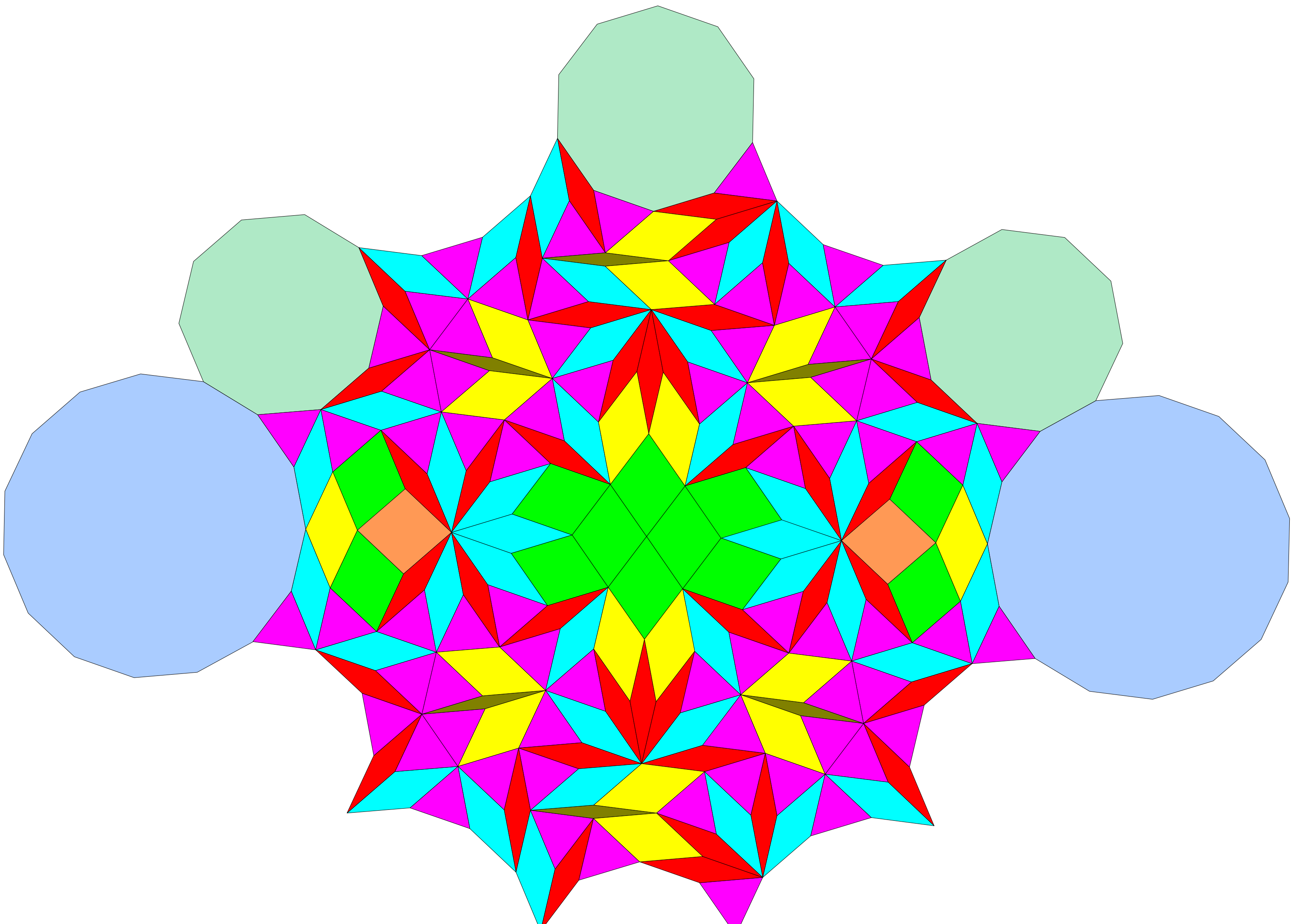

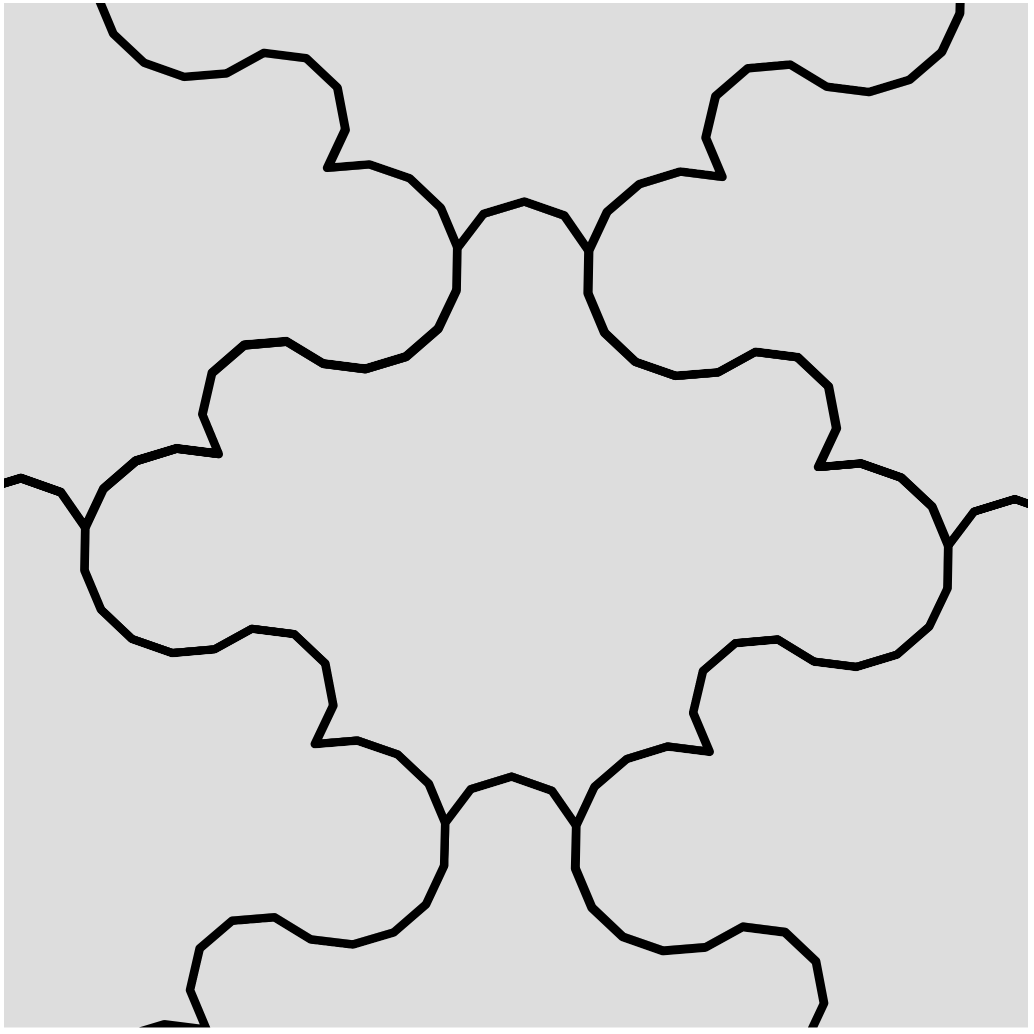

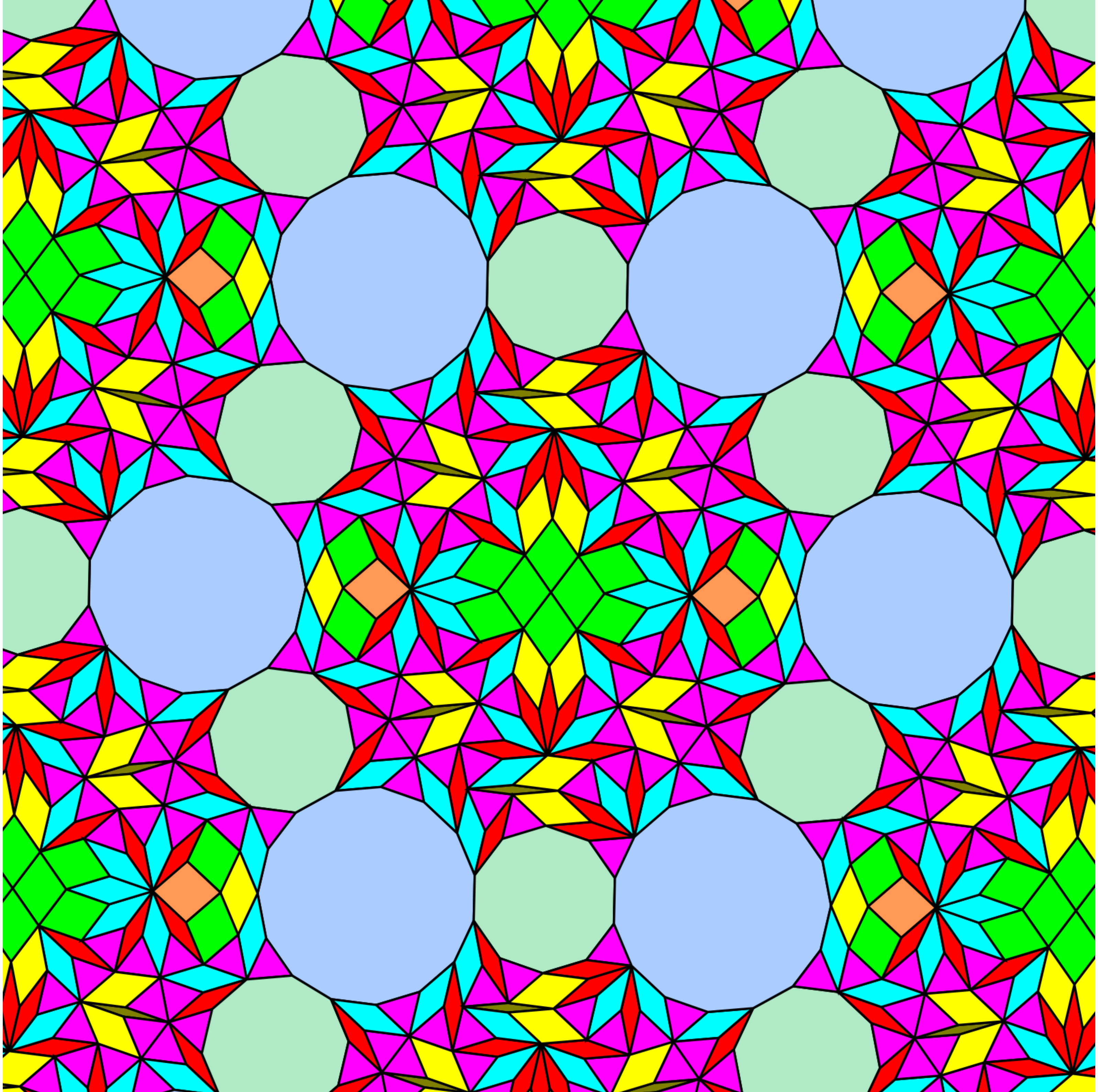

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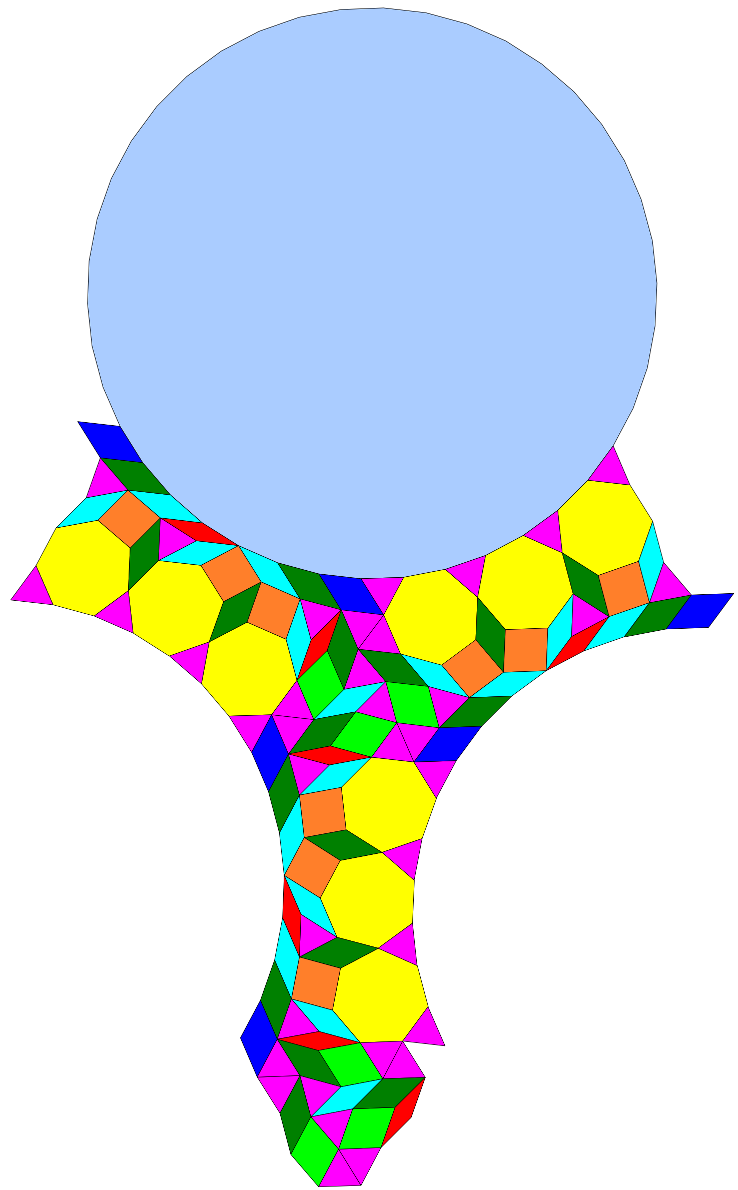
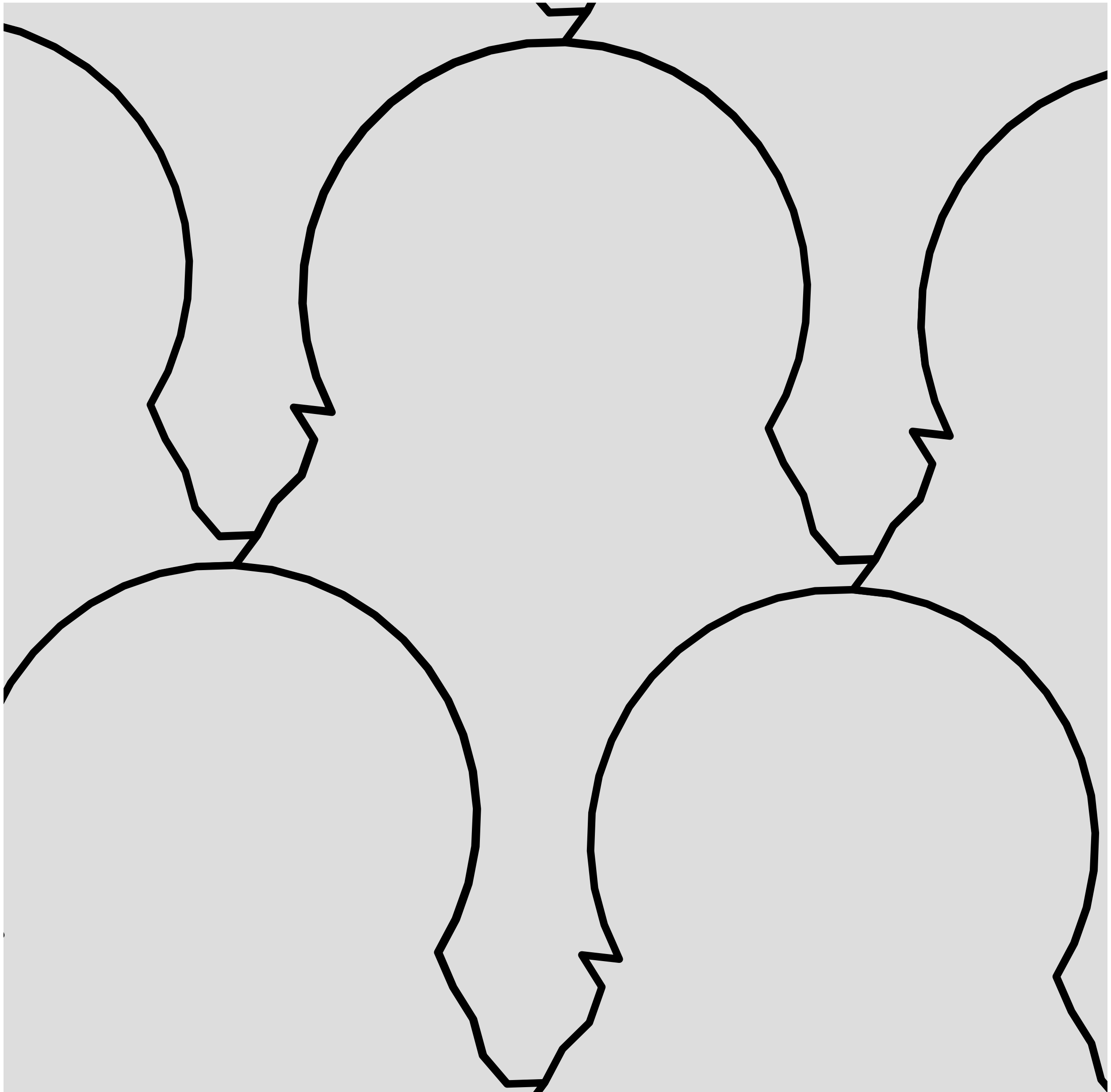
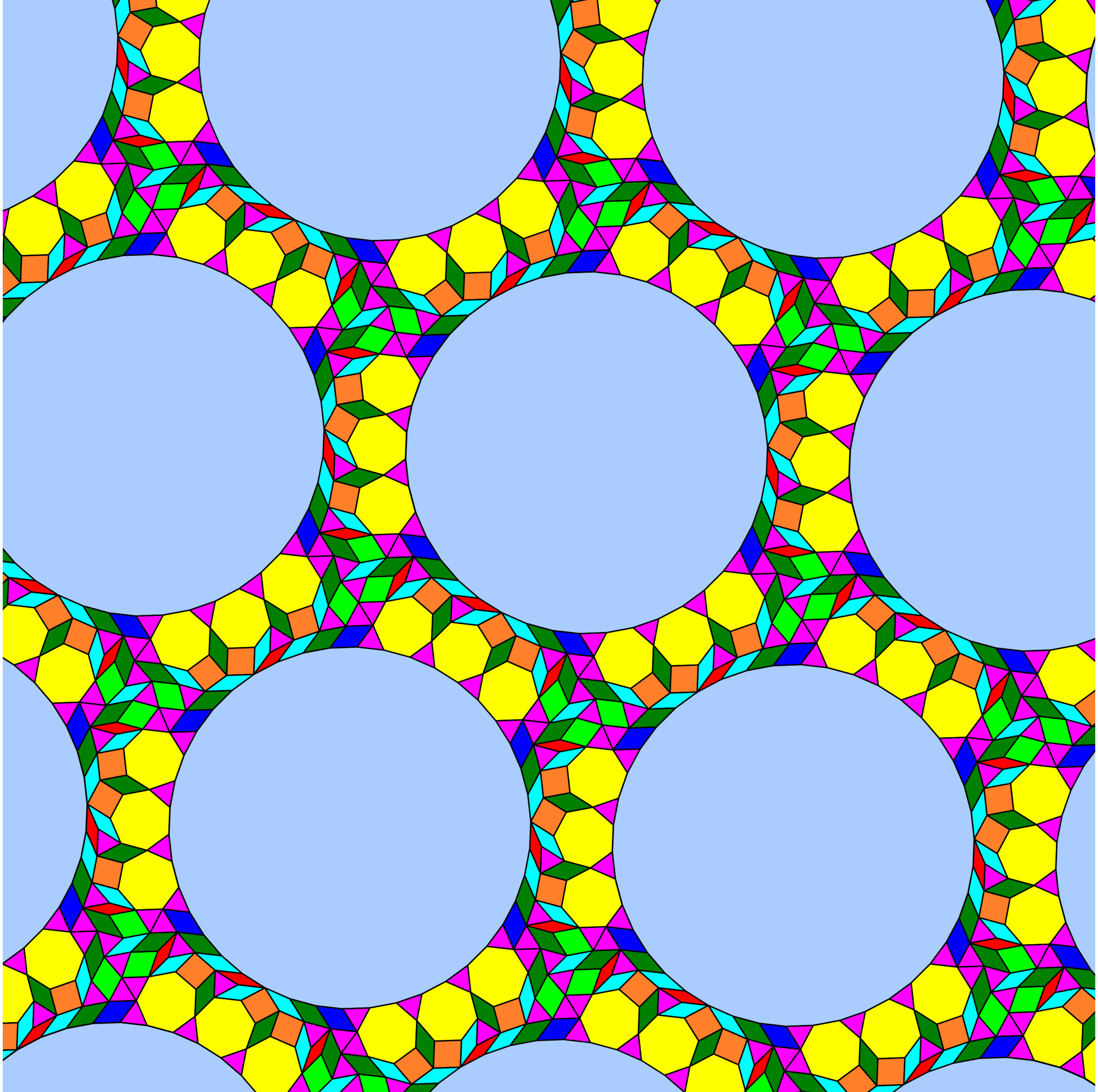
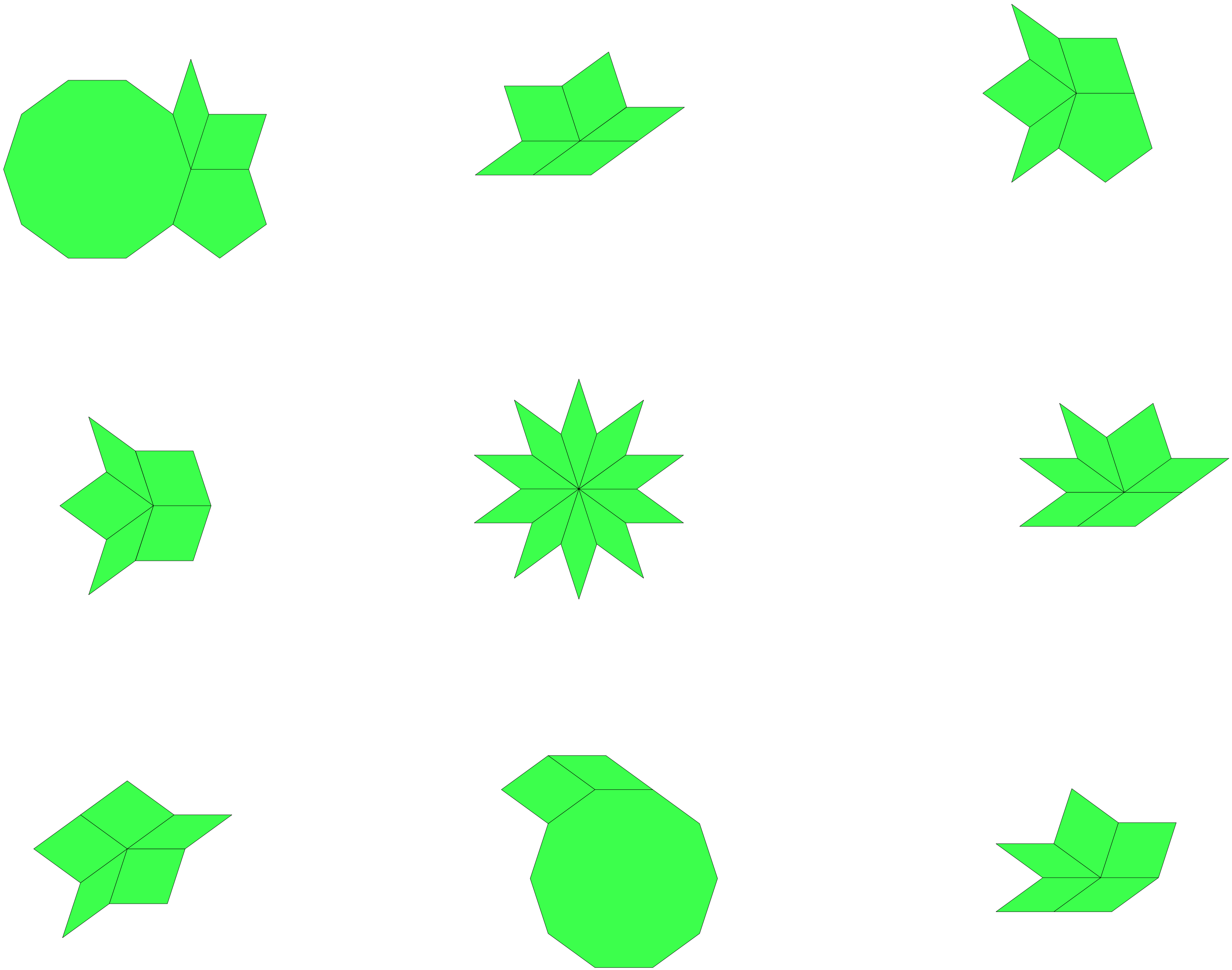




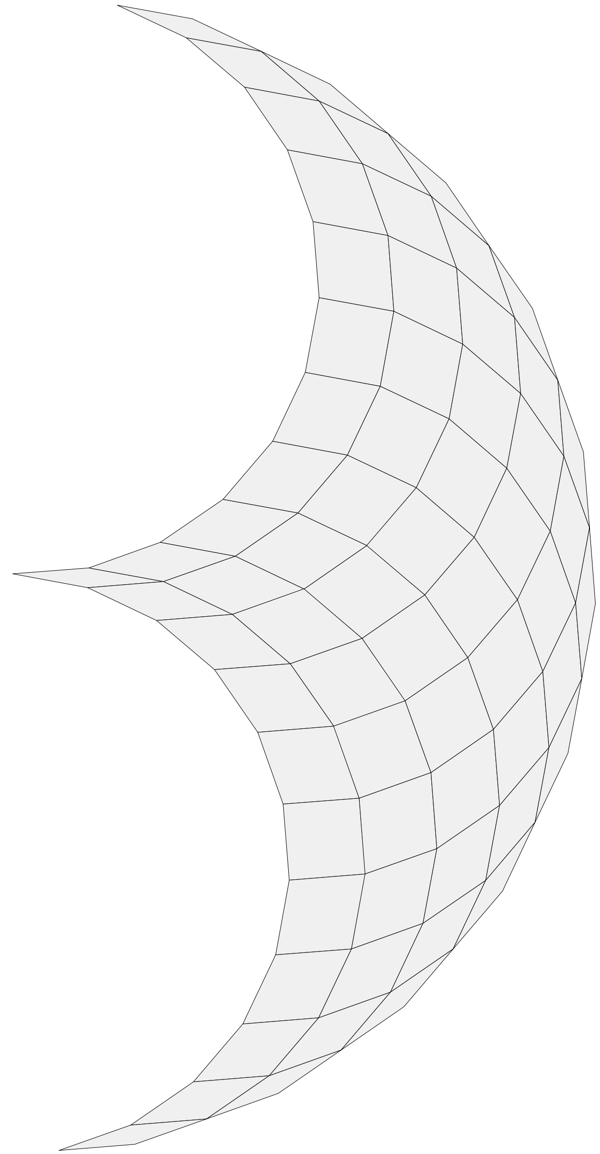


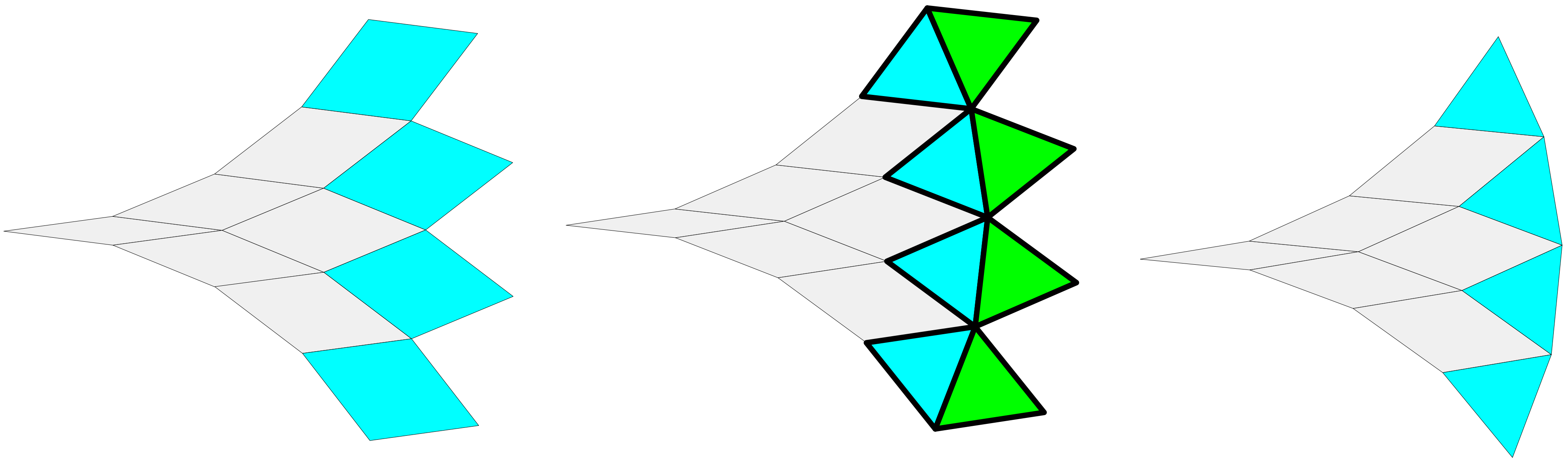


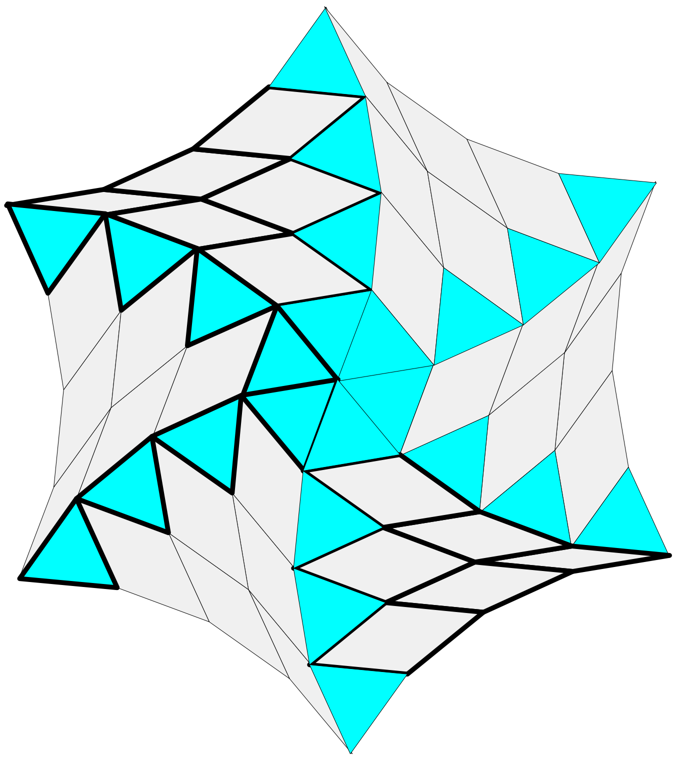
Two algorithms

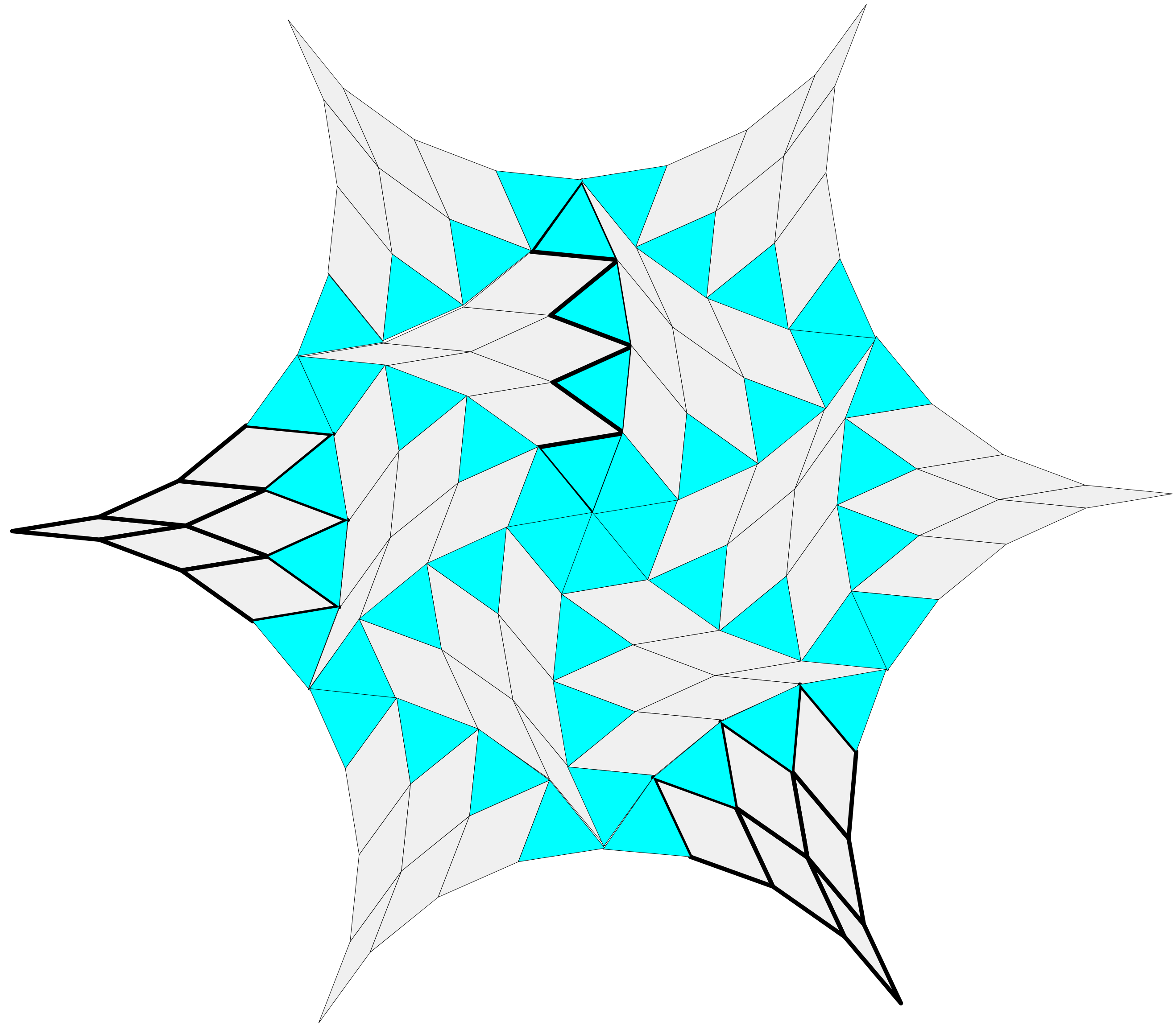
- Extend any regular polygon to a periodic tiling of the plane.

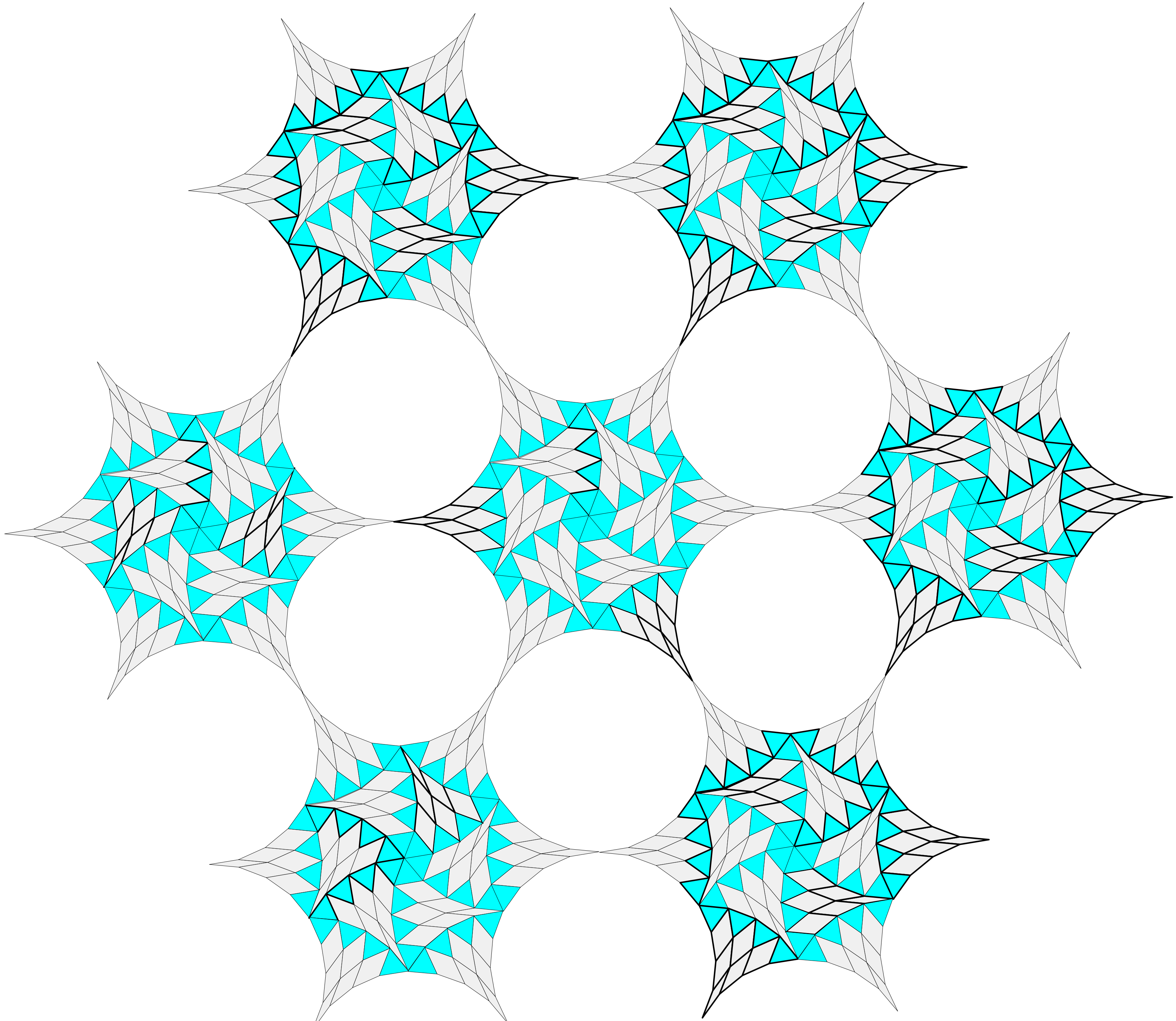
- Extend any vertex figure of regular polygons and rational rhombs to a regular polygon.





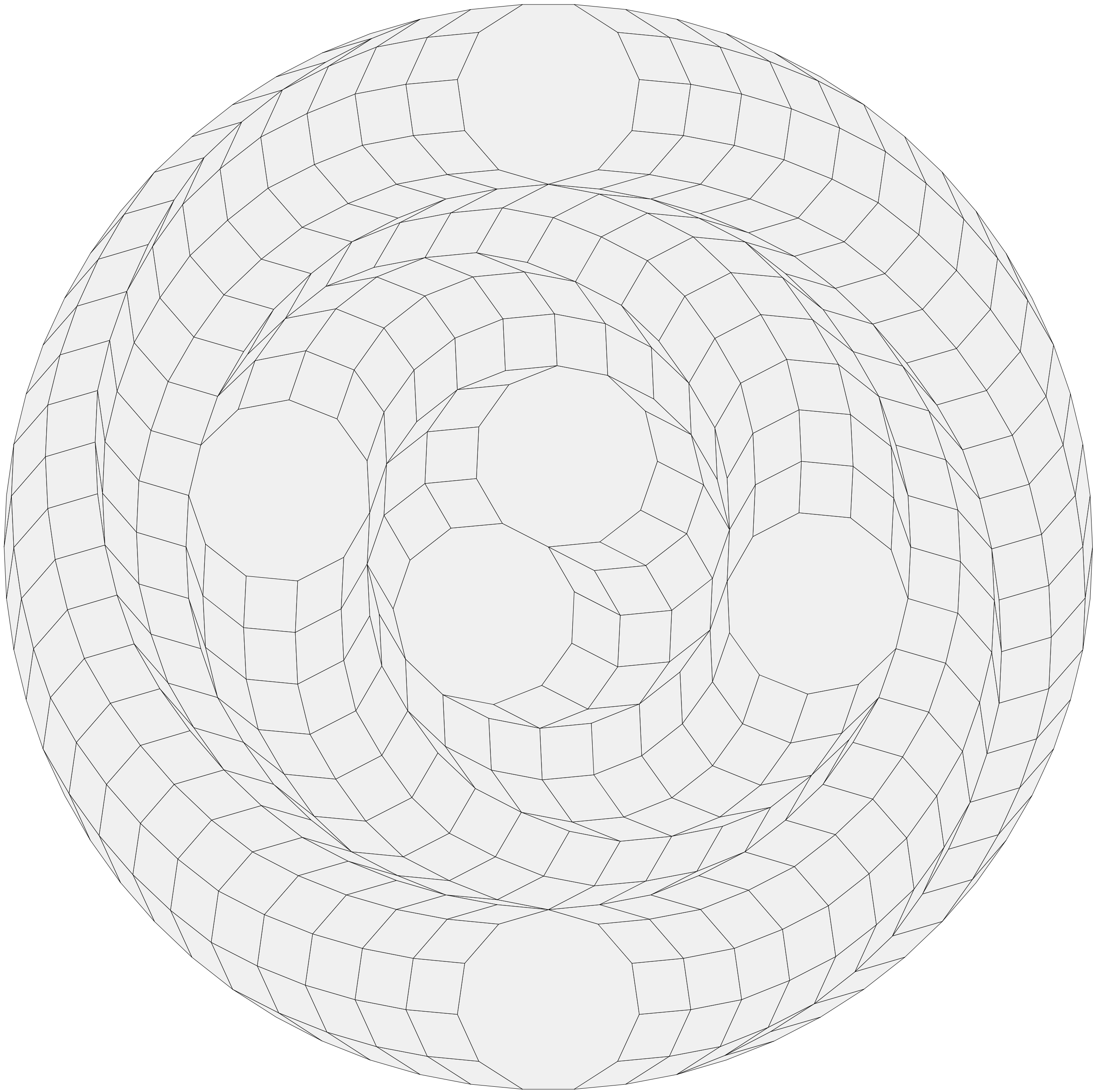


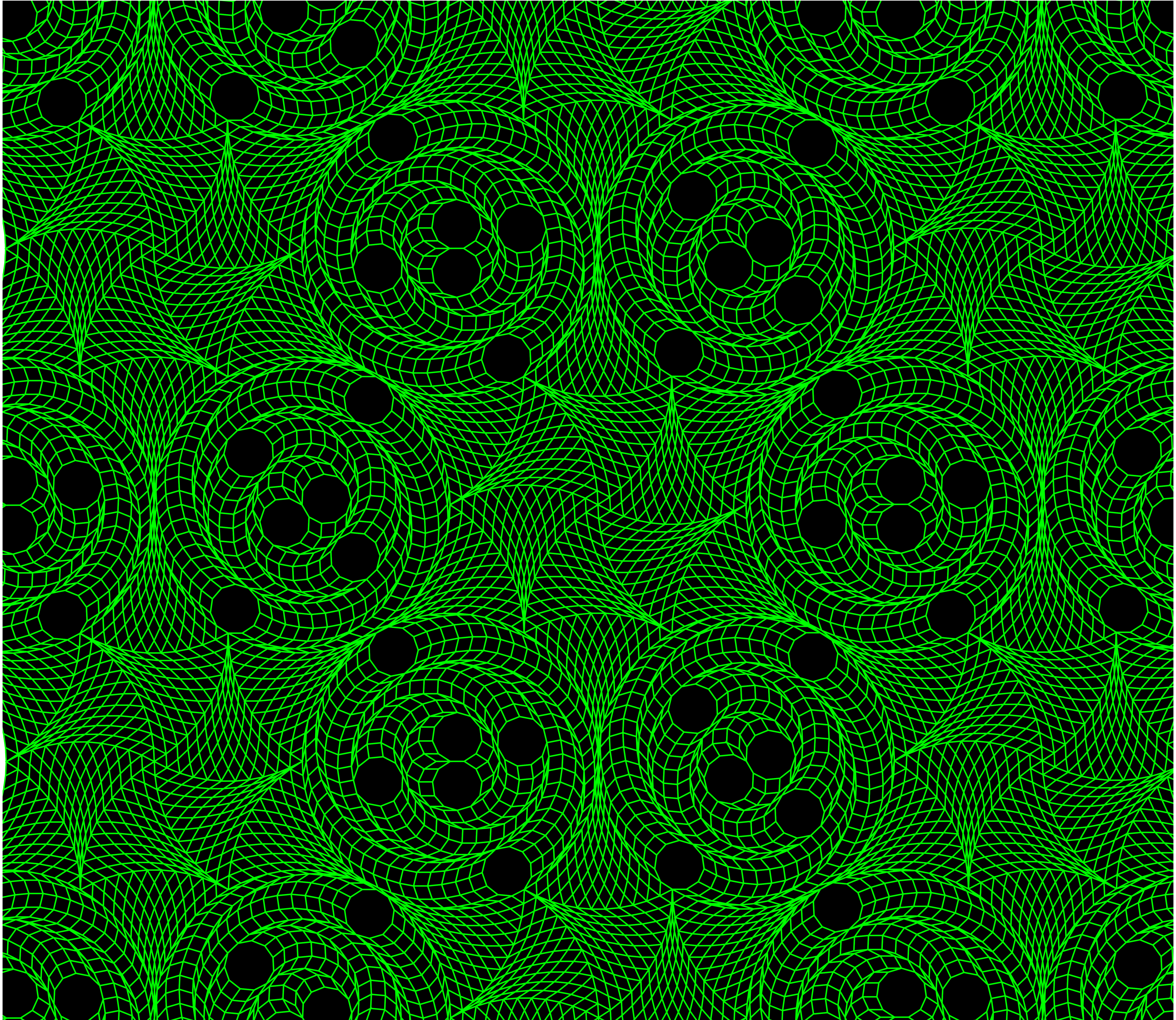


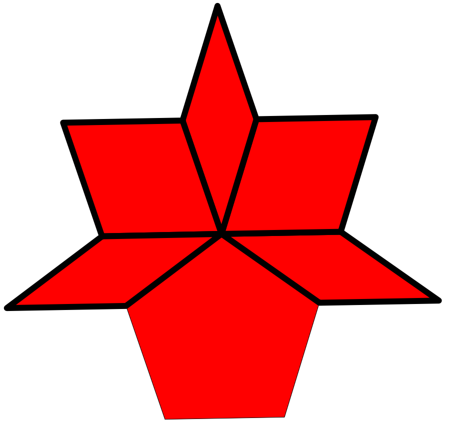


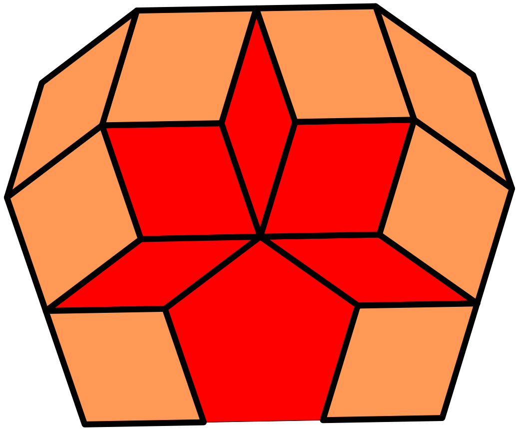
Decomposition algorithm

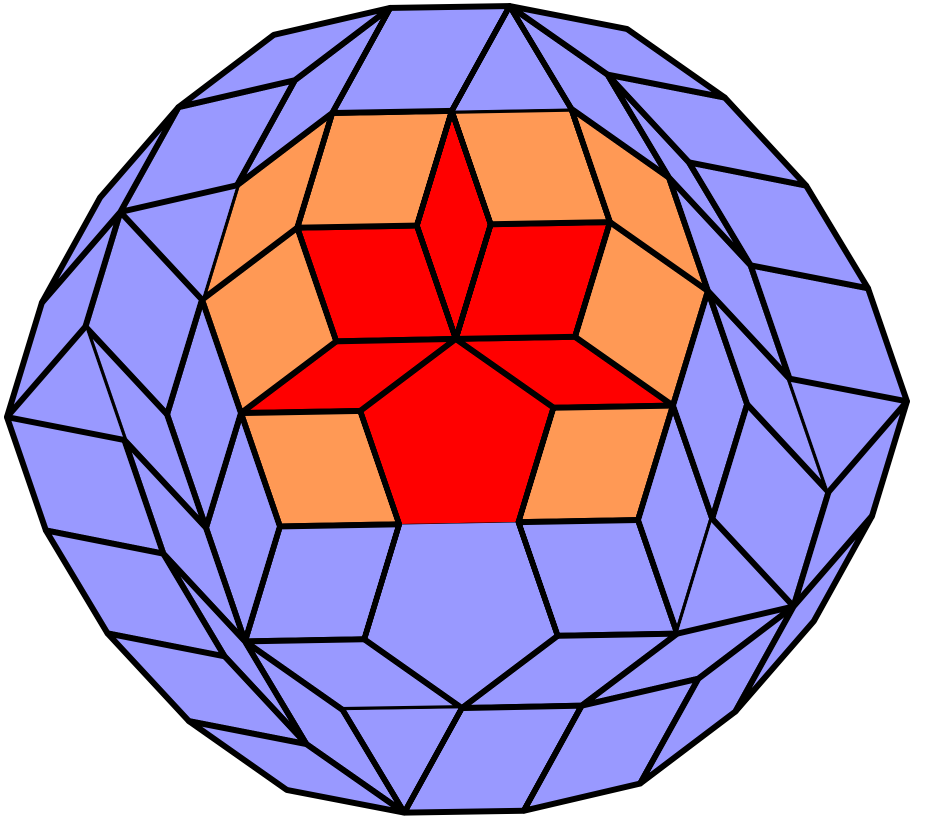
A regular polygon of composite order mn ($m > 1$, $n > 2$) can be decomposed into m n-gons and rational rhombs.

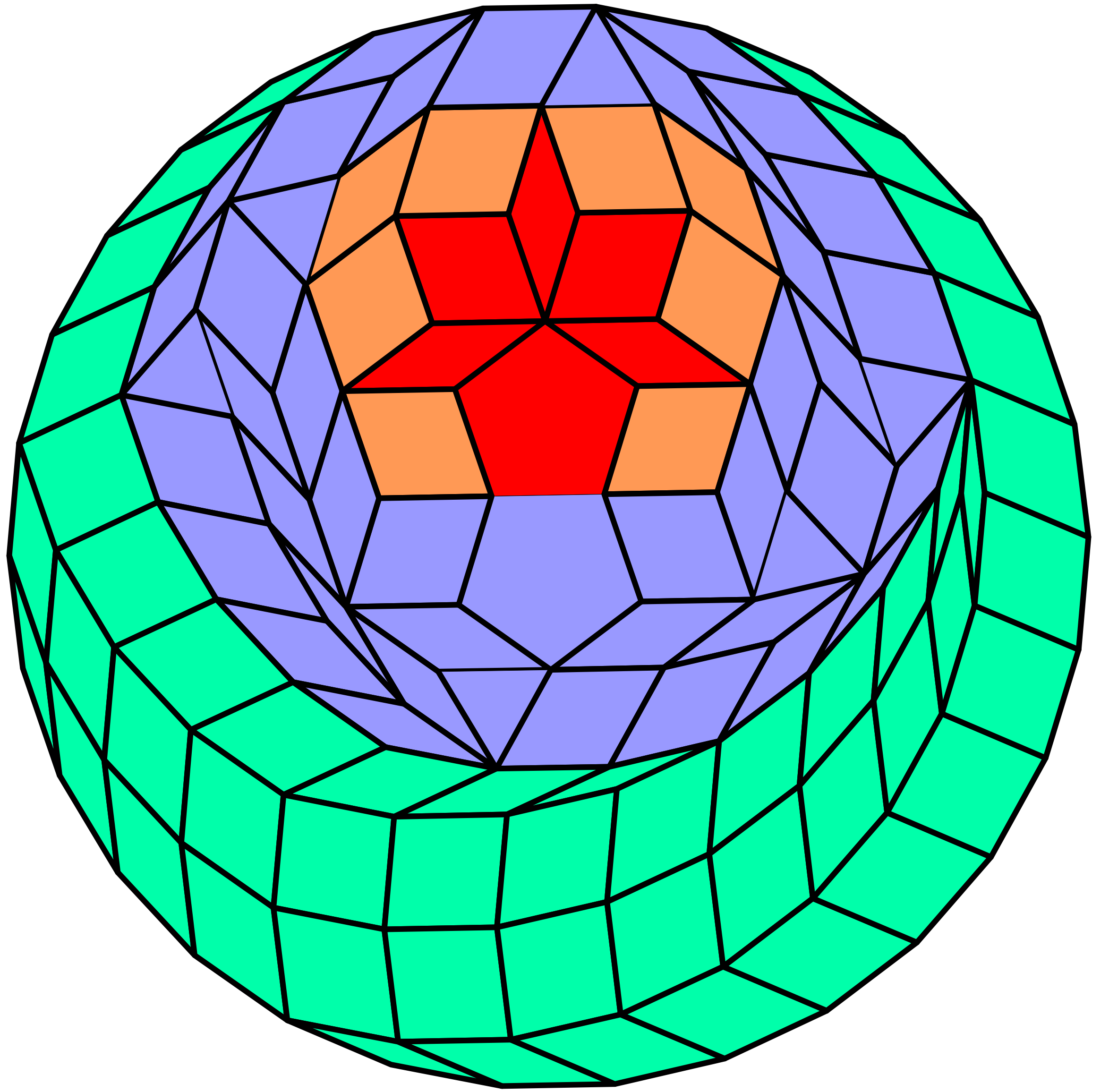


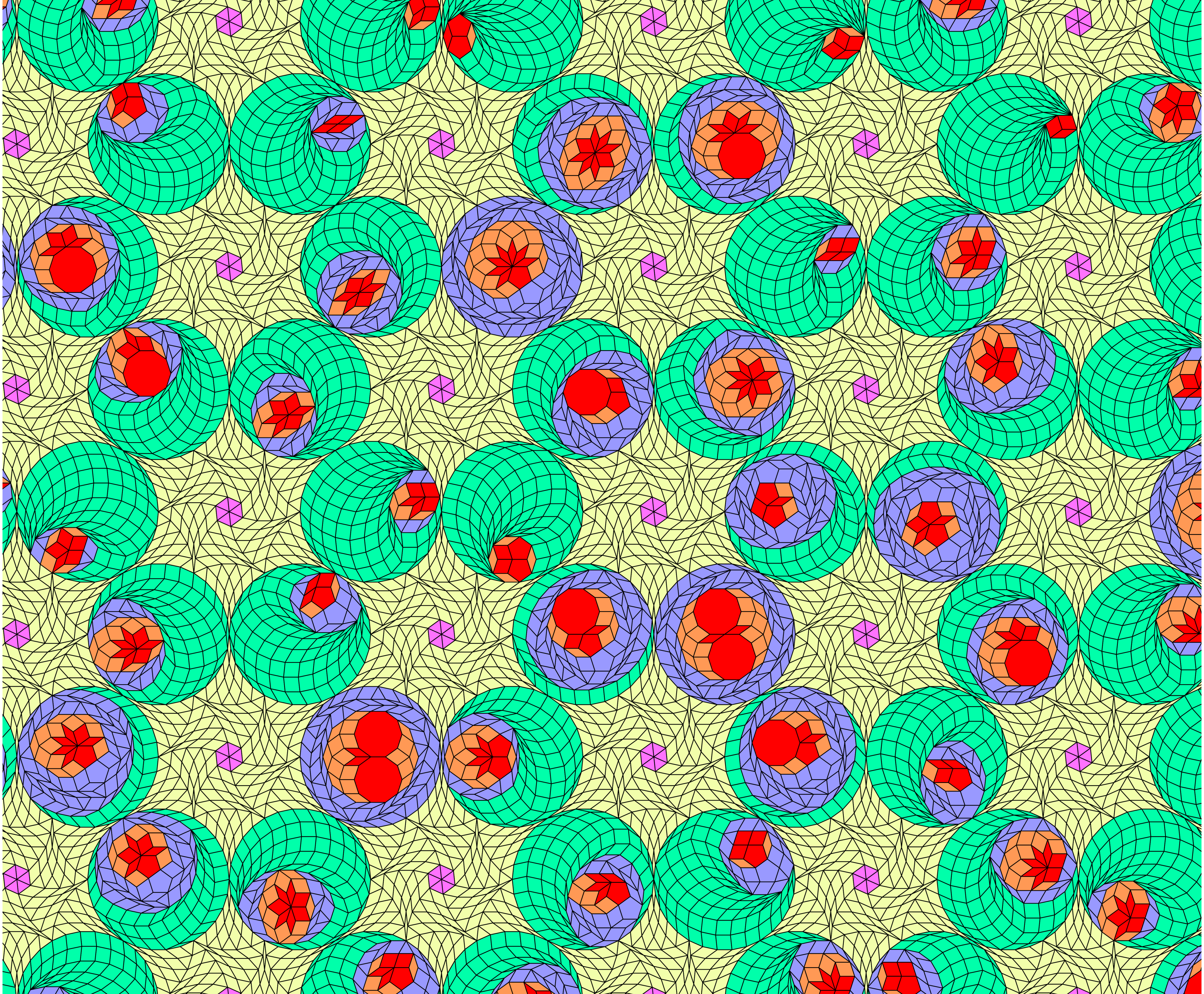


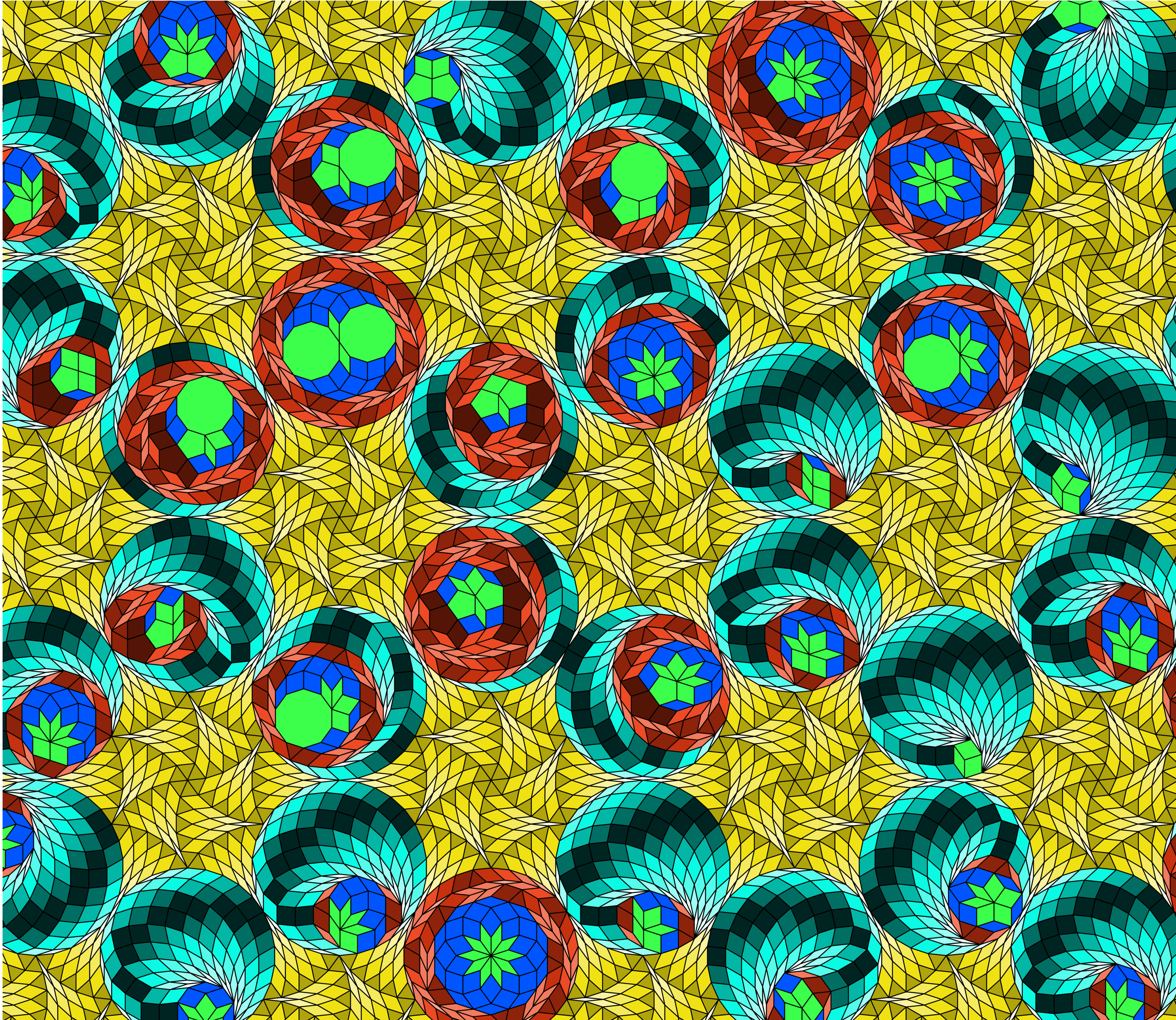


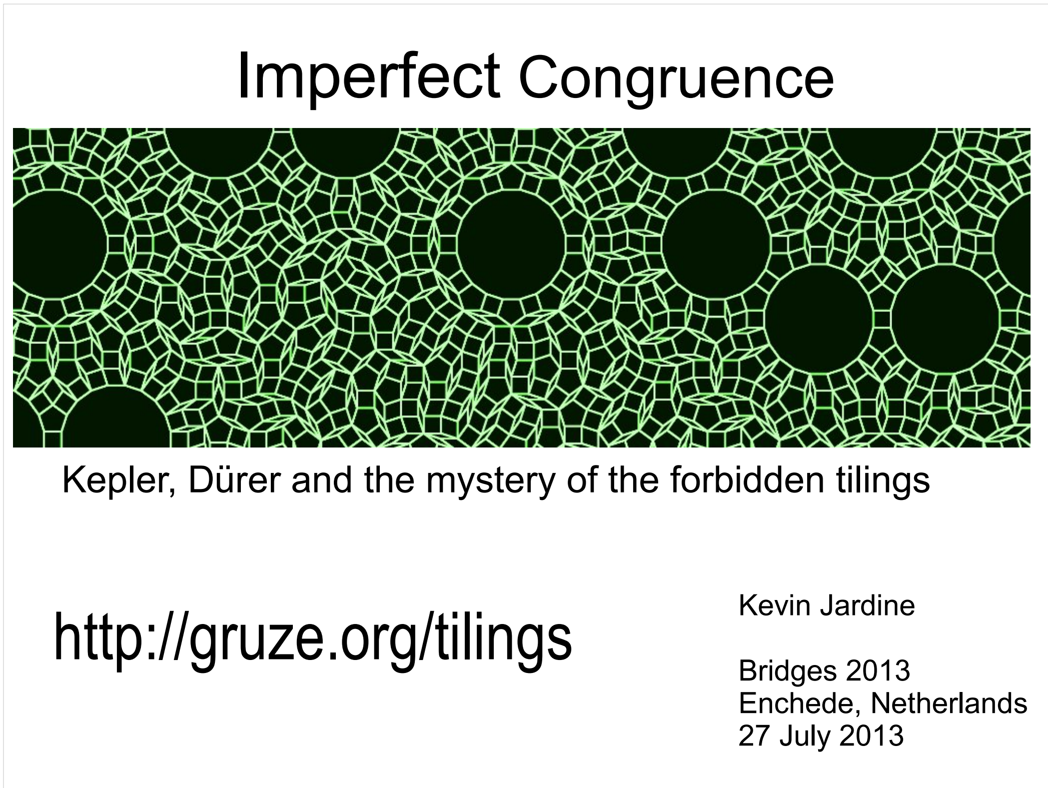






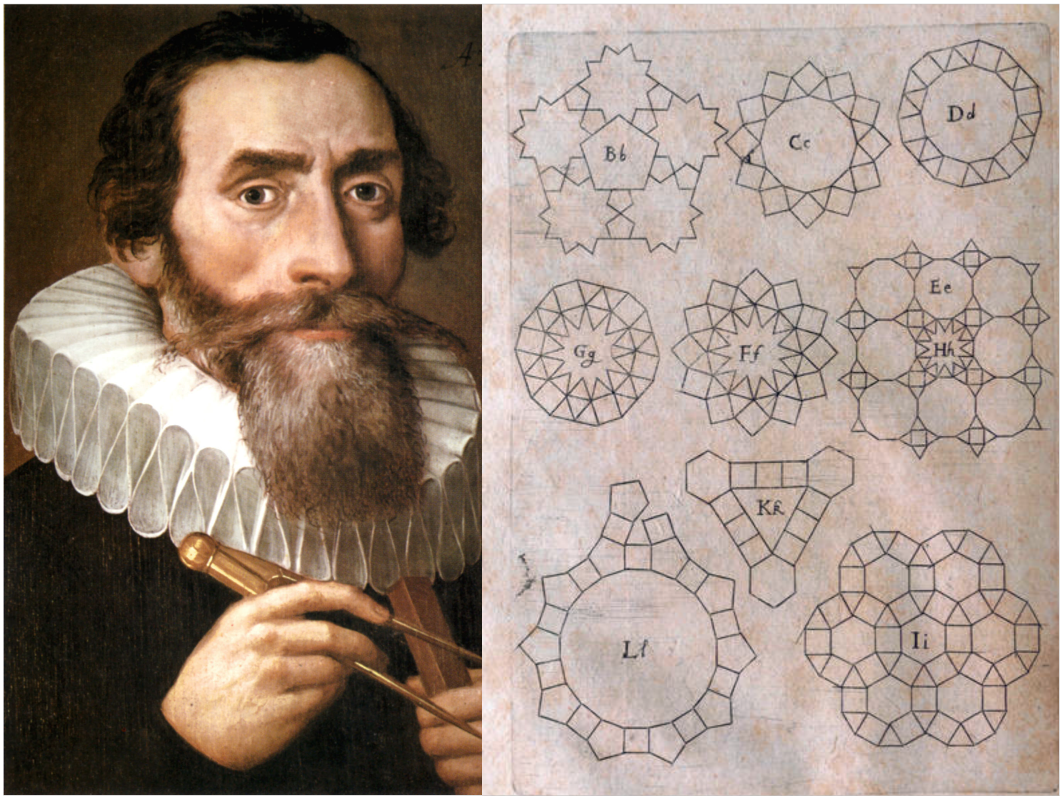






I'm Kevin Jardine, the creator of the Imperfect Congruence site at gruze.org/tilings.

This presentation is a kind of teaser for the (rather large) website as there is no way I could cover even a fraction of the content in a brief paper or presentation.

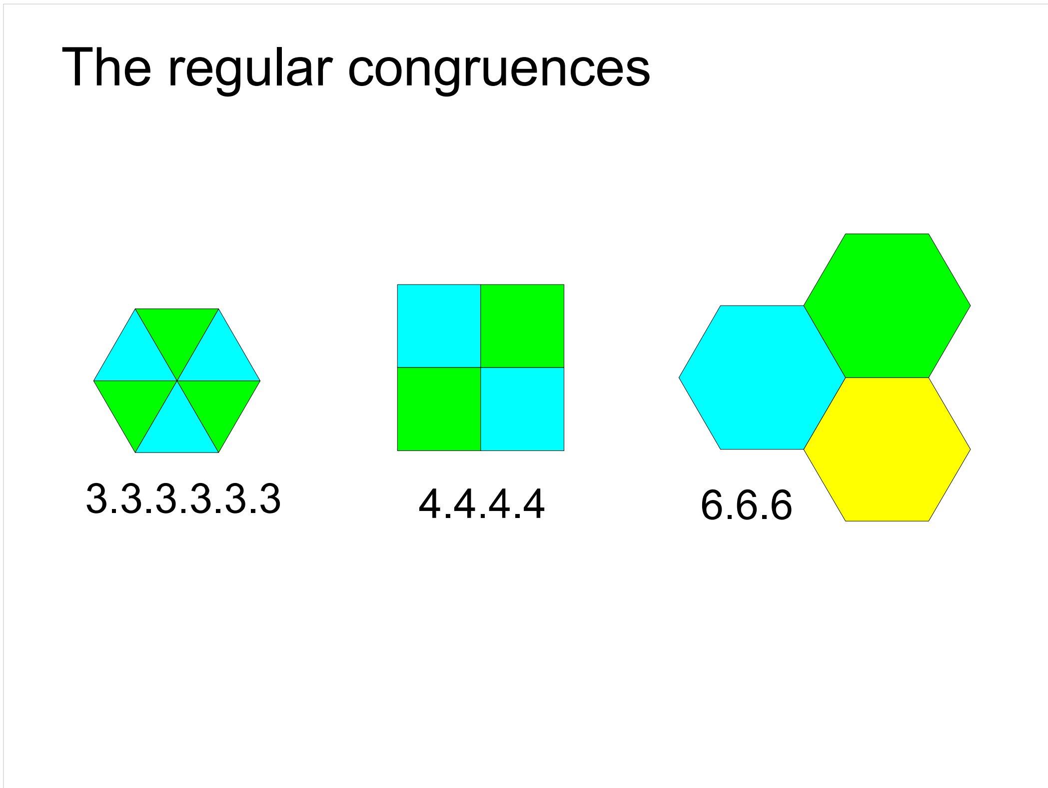


First of all, what is an imperfect congruence?

Johannes Kepler introduced the term in Chapter II of his 1619 book The Harmony of the Worlds. Chapter II contains the first organised presentation on tiling regular polygons in the plane. That chapter also discusses several other kinds of tiling but I won't discuss those today.

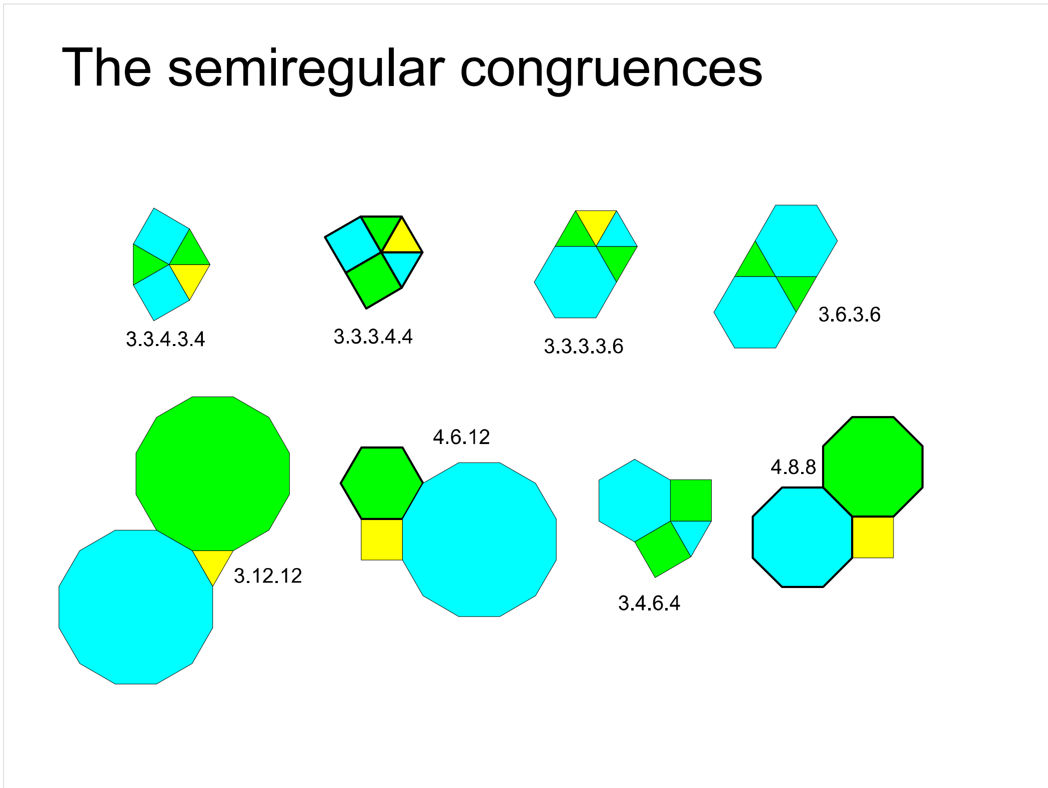
Kepler defined a congruence as an arrangement of regular polygons around a central vertex. Today this is more often called a vertex figure.

It turns out that there are only 21 ways regular polygons can be arranged around a vertex. These 21 congruences fall naturally into 4 groups.



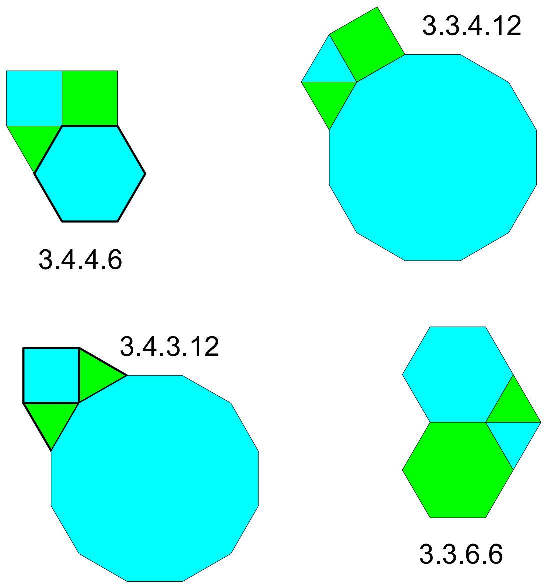
Kepler defined a perfect congruence as one that can be extended to a periodic tiling of the plane which has the same congruence at each vertex. Today these are usually called the uniform tilings.

The regular congruences are the most perfect congruences as they contain the same polygons. There are only 3 of these formed by the equilateral triangle, square and hexagon.

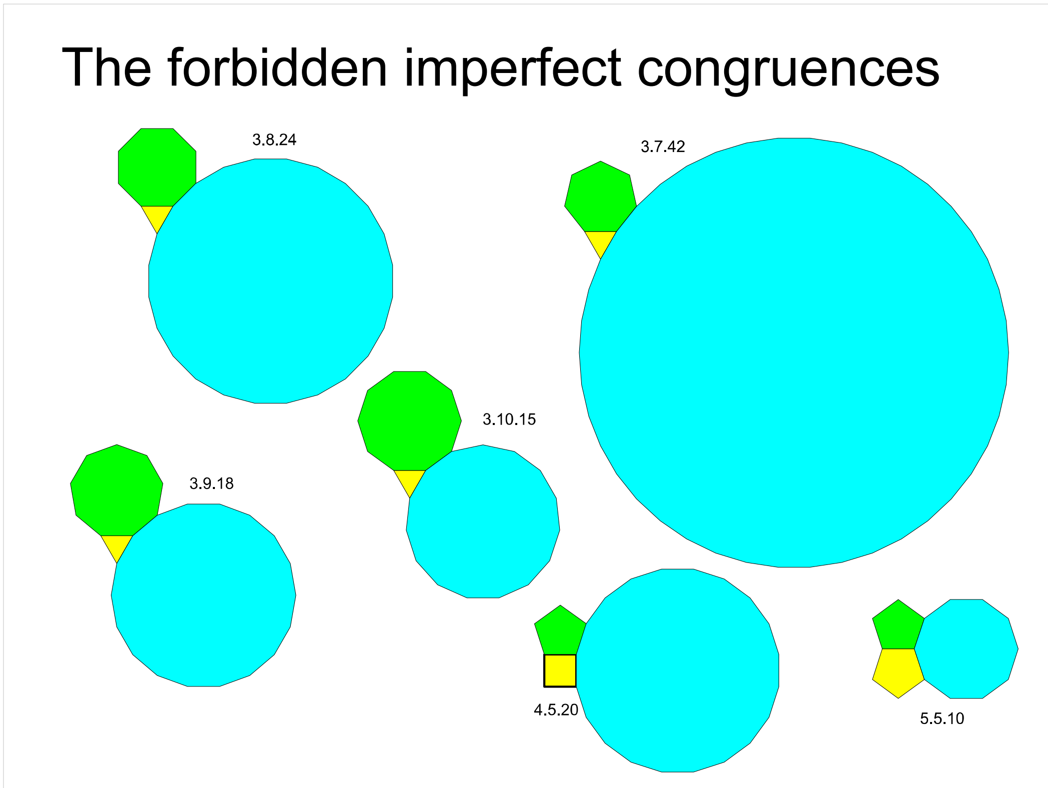


In addition to the regular congruences, there are also 8 semiregular congruences. These can also be extended to uniform tilings but consist of more than one type of polygon. You often see floor tilings based on the vertex figure consisting of squares and octagons. The other figures are also sometimes used but much less often.

The extendable imperfect congruences



The are two kinds of imperfect congruences. These four can be extended to periodic tilings but require introducing a different congruence so the arrangement of polygons is not the same at every point.



The second type are the forbidden imperfect congruences. They are forbidden because it is fairly easy to prove that it is impossible to extend them to periodic tilings of the plane, even by adding other congruences.

This is disappointing really, because the forbidden vertex figures contain pentagons, heptagons, nonagons, and even a 42-gon. As a result, regular polygon tilings are fairly simple.

I originally created the Imperfect Congruence website to look at ways to extend these vertex figures to periodic tilings of the plane.

How can we do this? Obviously something more than regular polygons is required.

$$2x - 1 = 0$$

It is a common situation in mathematics to have to extend a system to solve problems possed by that system.

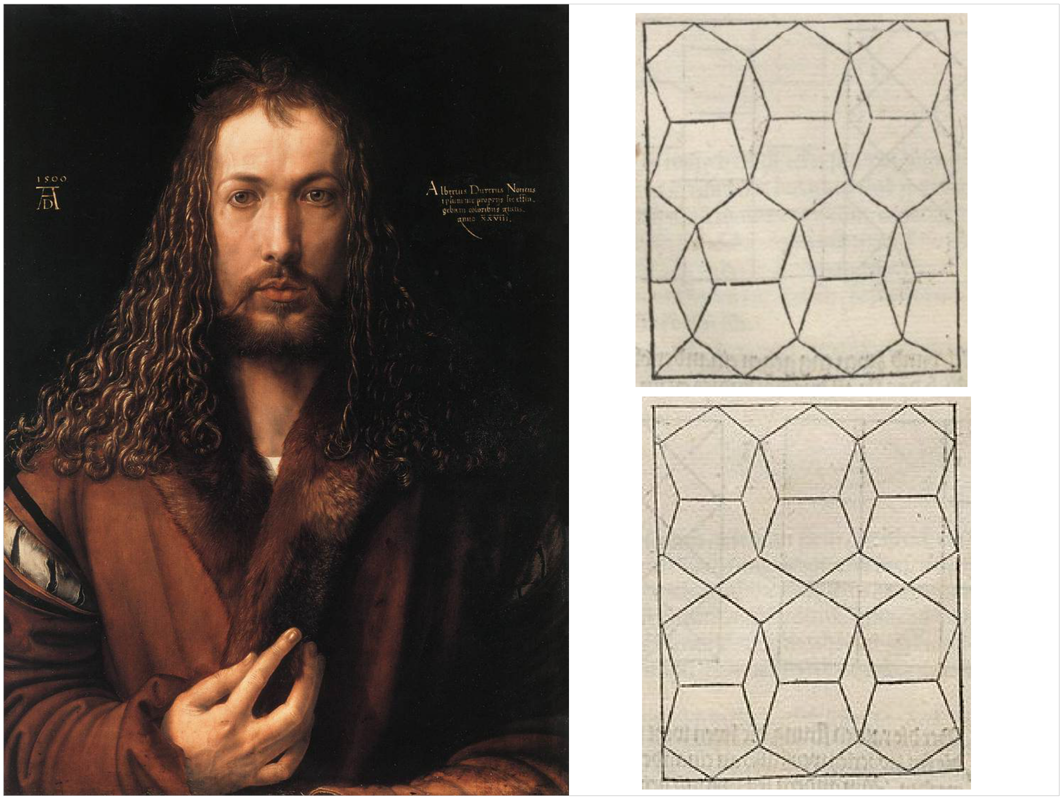
Consider polynomial equations with integer coefficients. This one requires introducing the rational number 1/2 to solve.

$$x^2 - 2 = 0$$

This one requires the irrational square root of 2.

$$x^2 + 1 = 0$$

This one requires a complex number, the square root of -1.



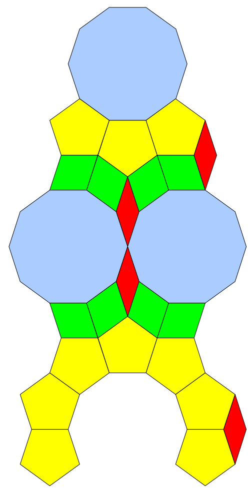
What shapes do we need to add to allow the forbidden congruences to be extended to periodic tilings?

The artist Albrecht Durer provided a strong hint in a treatise he published almost 100 years before Kepler. He showed several examples of tilings including pentagons. He is able to get these to tile by introducing rhombuses or as I'll call them rhombs, into the prototile sets.

Here are two examples.

A rhomb has four equal sides like a square except with two distinct angles. If one angle is alpha, the other is always pi - alpha. In my talk from now on I'll restrict the discussion to rational rhombs, whose internal angles are a rational multiple of pi.

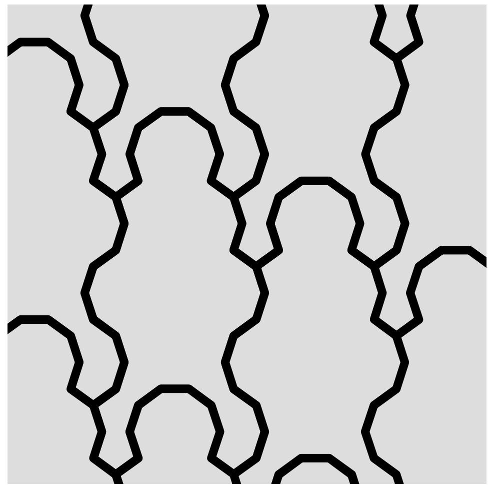
5.5.10



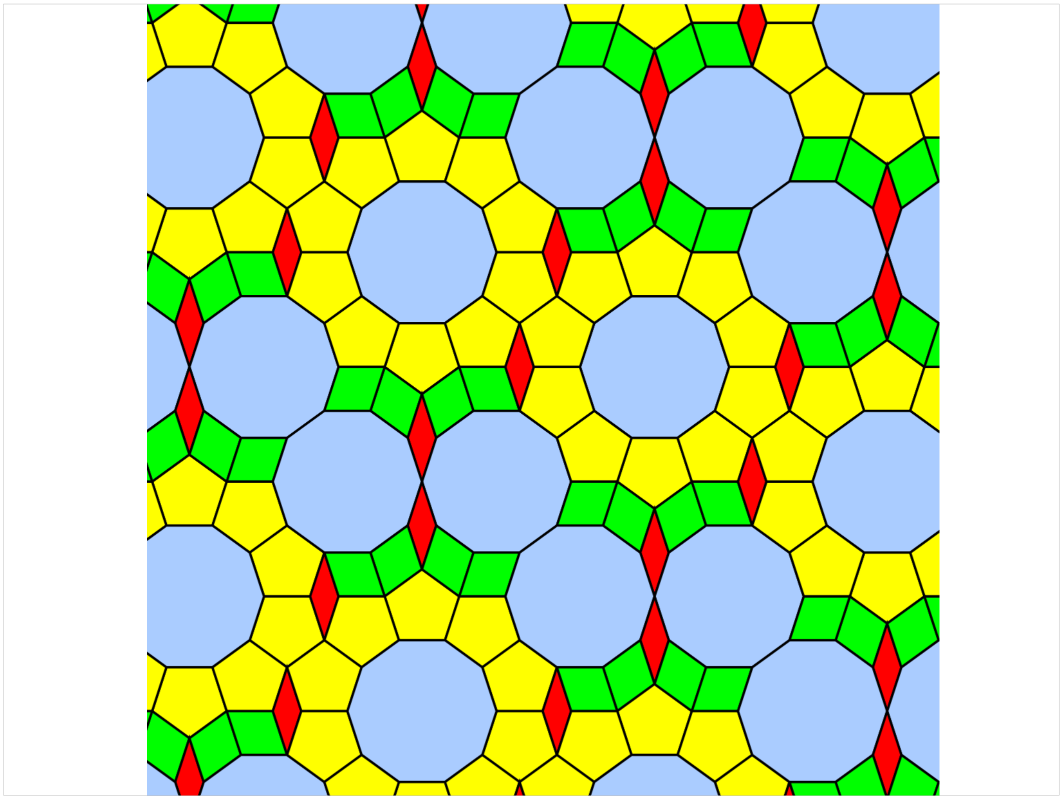
As it turns out, all of the forbidden congruences can be extended to periodic tilings with the help of a small number of rational rhombs. I can prove this by extending each forbidden congruence to a small patch called a translational unit which can be repeated to tile the plane.

There are many ways to do this. I've chosen translational units that have some aesthetic appeal. They are not the simplest possible. There are a number of others on my website.

Here is the one for the figure that has two pentagons and a decagon.

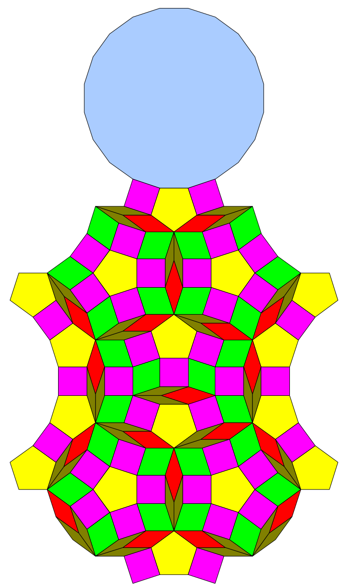


Here is how to repeat the unit.

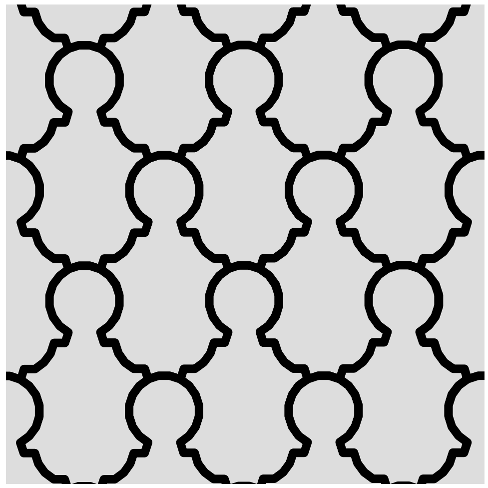


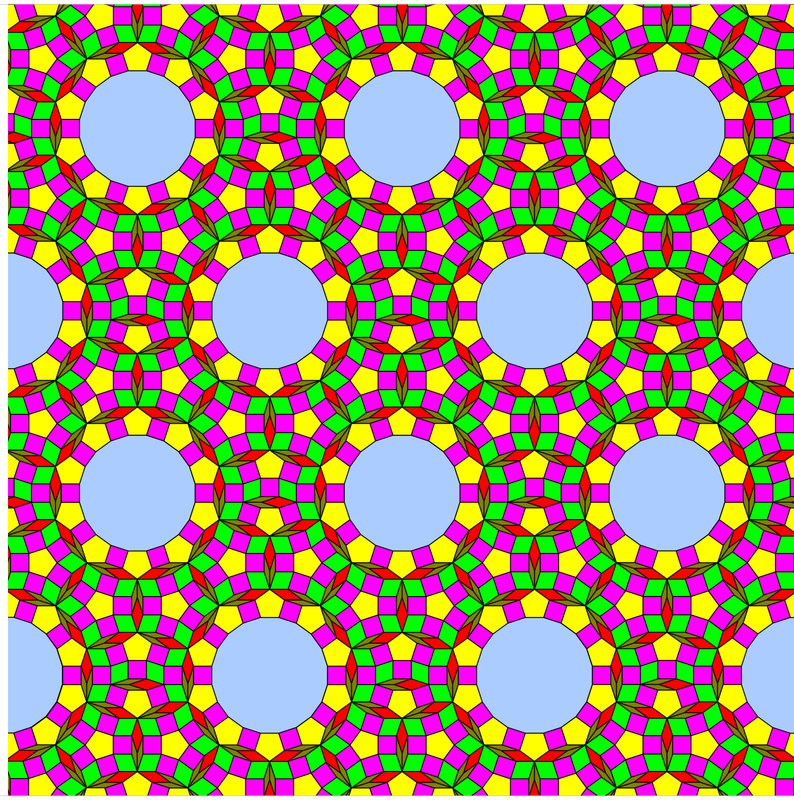
Here is a detail of the resulting tiling.

4.5.20



I'll show the same results for the other five forbidden figures.





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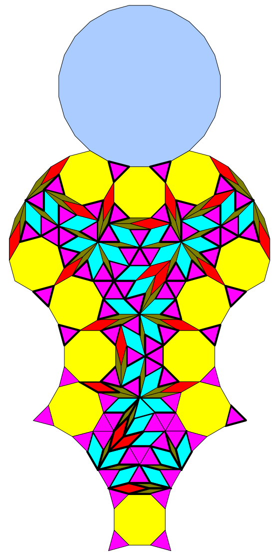

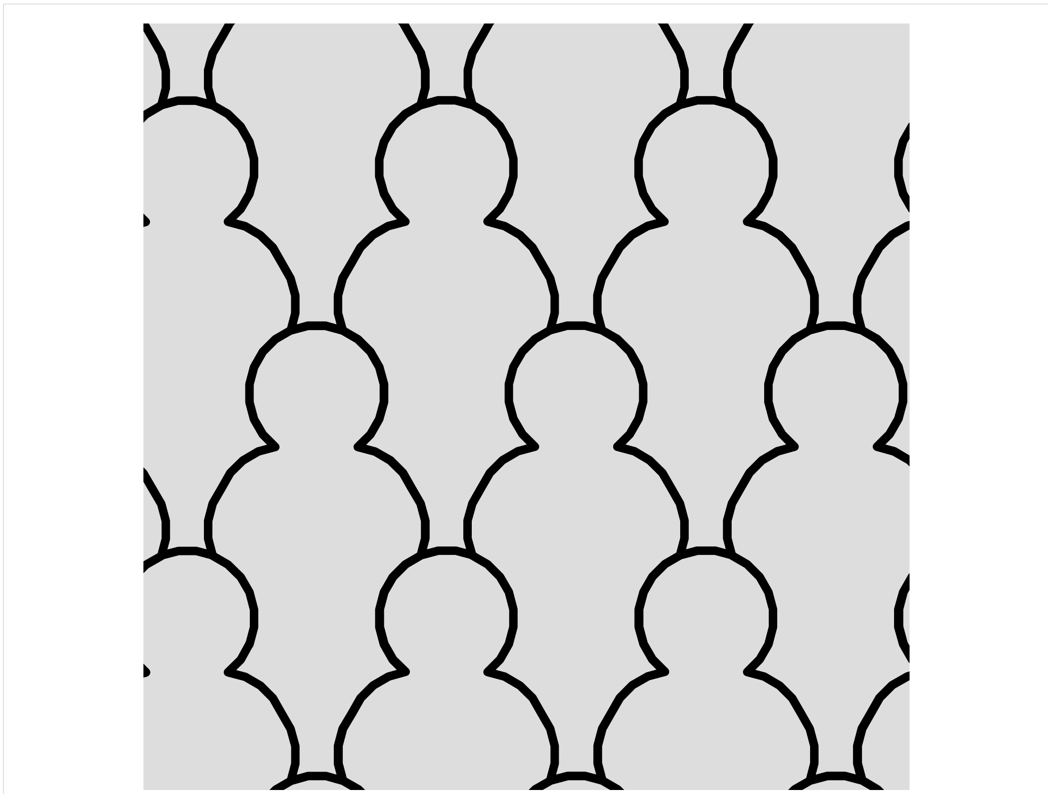

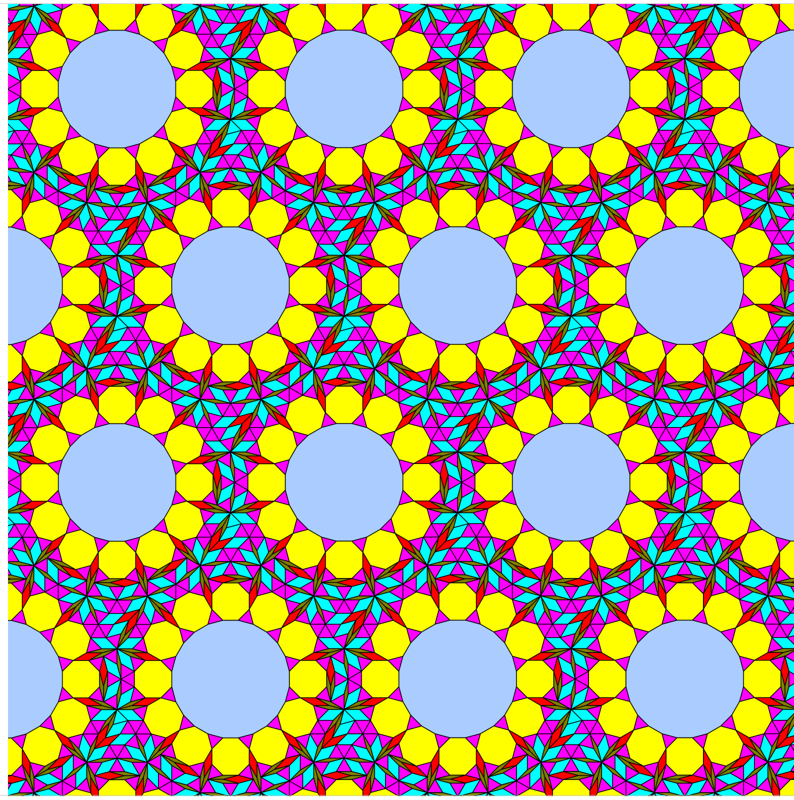

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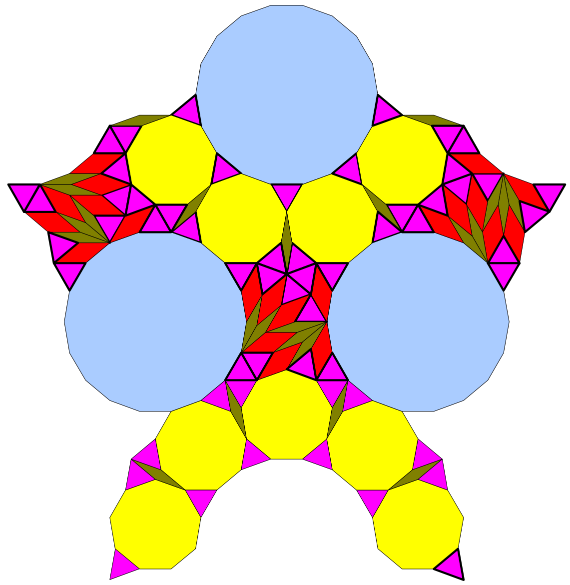
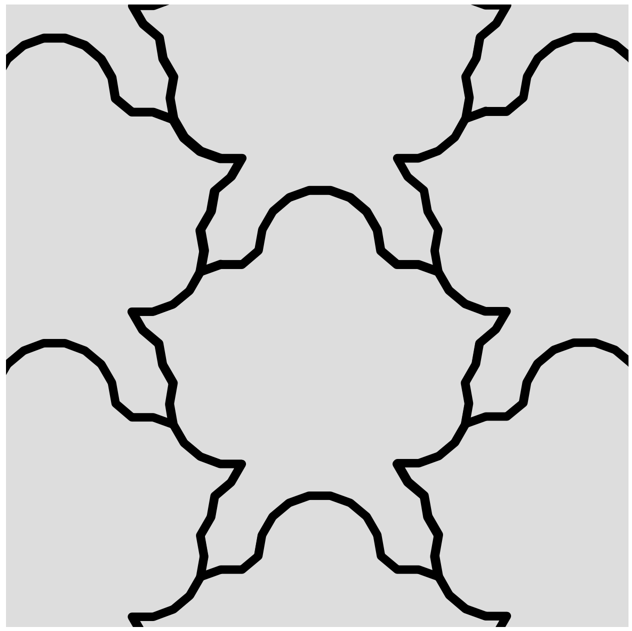
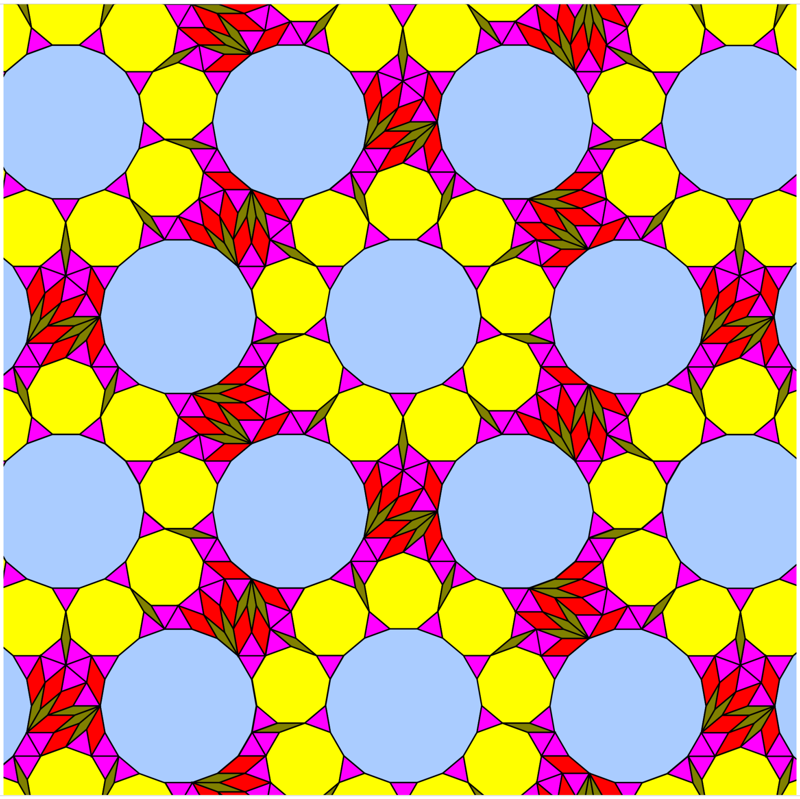
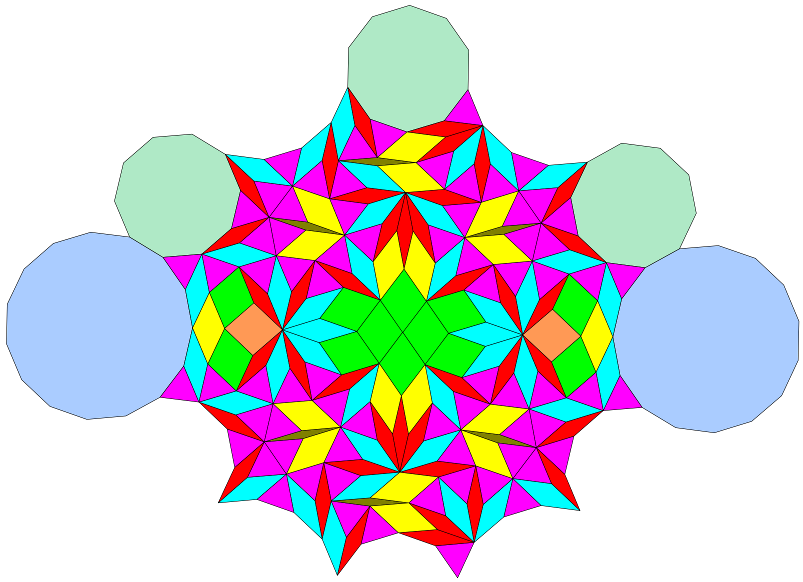
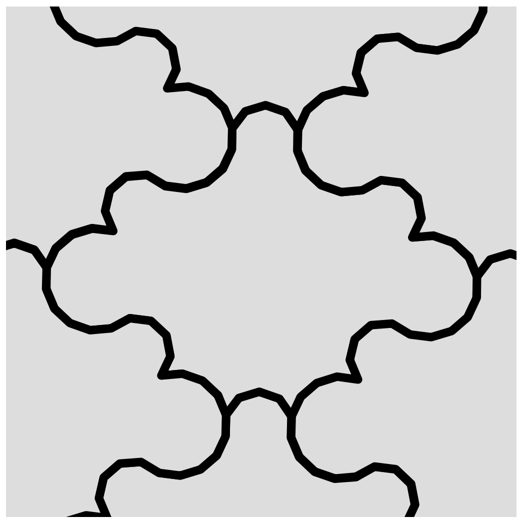


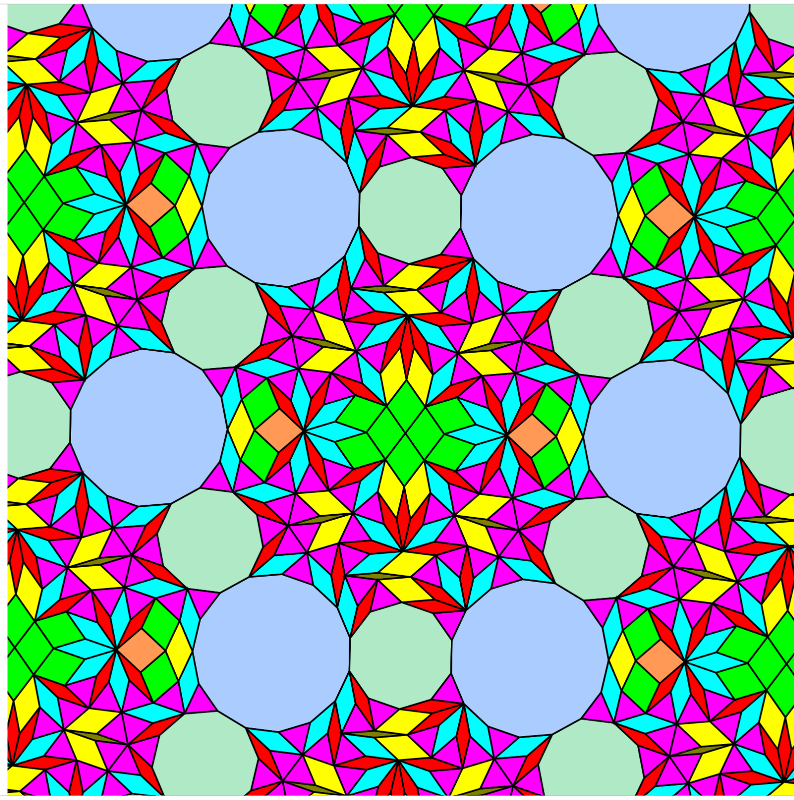




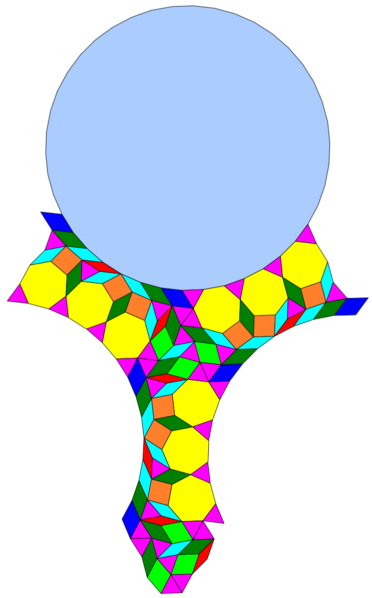
3.10.15

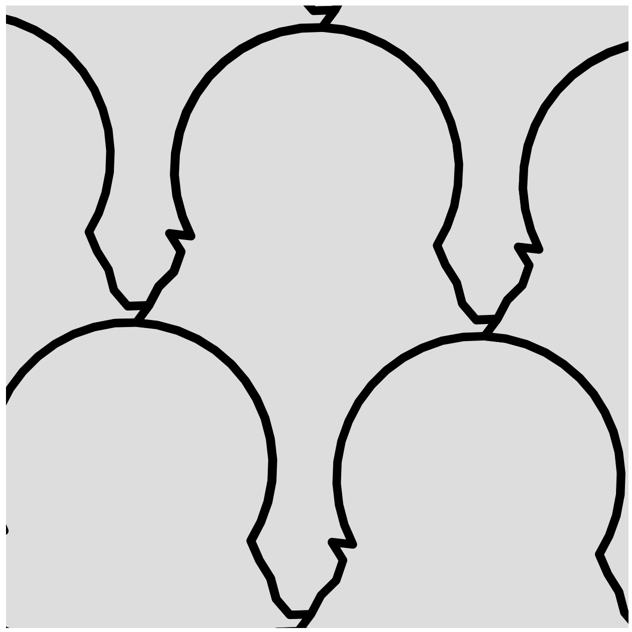


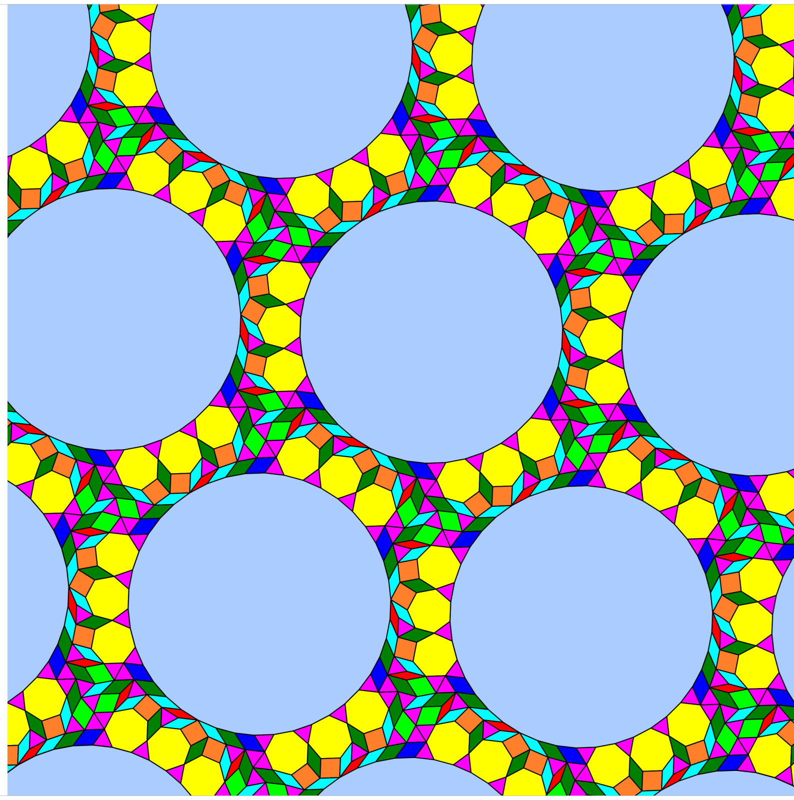


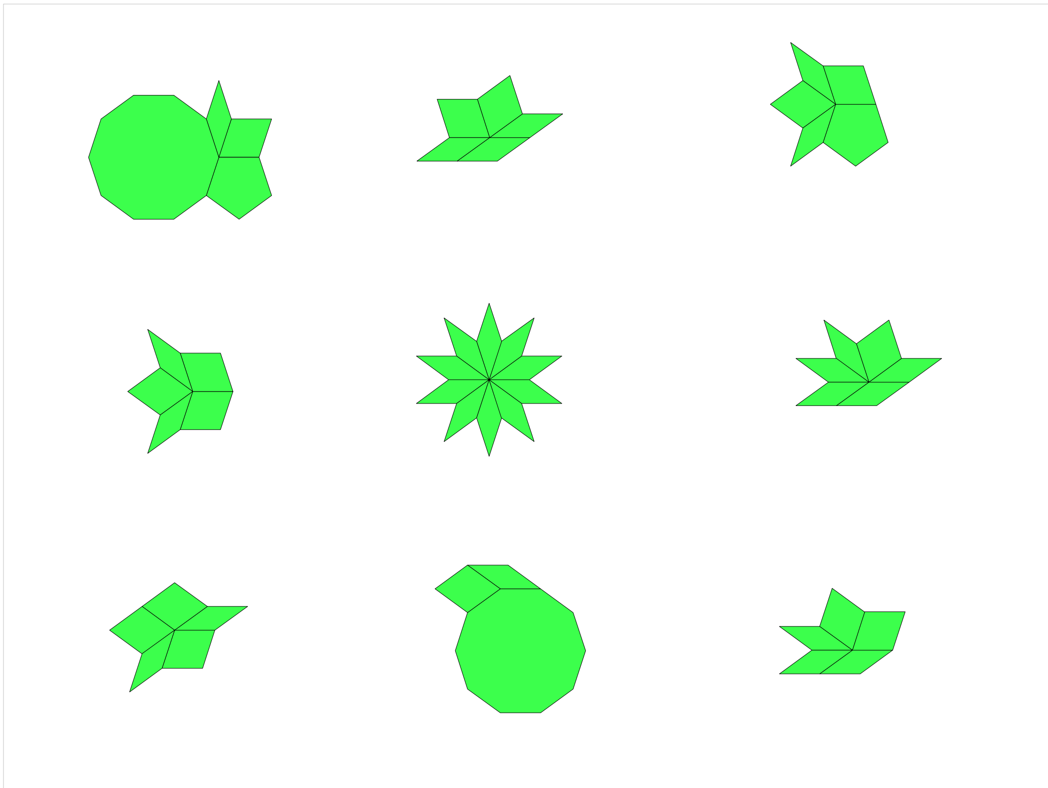


3.7.42









I've rushed through this a bit because I wanted to get to the mathematically intriguing part.

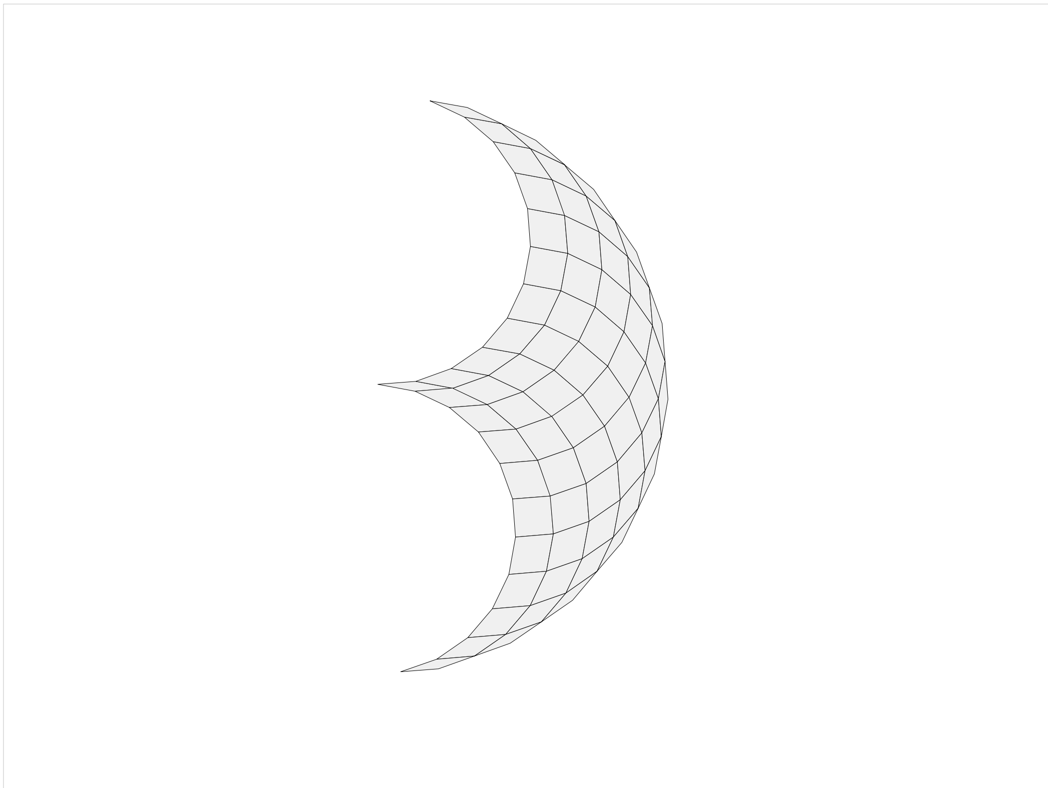
The interesting thing about complex numbers is that the fundamental theorem of algebra says that any nondegenerate polynomial equation has at least one root in the complex numbers. This is true even if the coefficients of the equations are allowed to be complex numbers as well. In this sense the complex numbers form a complete system.

It appears that the set of reqular polygons and rational rhombs is also complete in the sense that any vertex figure of regular polygons and rational rhombs can be extended to a periodic tiling of the plane using only regular polygons and rational rhombs. This is not so easy to prove as there are an infinite number of such figures. Here are a few.

Two algorithms

- Extend any regular polygon to a periodic tiling of the plane.
- Extend any vertex figure of regular polygons and rational rhombs to a regular polygon.

I don't yet have a conclusive proof of this completeness. What I do have a couple of algorithms that appear to have some very interesting properties.



First, I'll say something about extending any regular polygon to a periodic tiling of the plane.

Firs let's define a rhombic fan of order m,

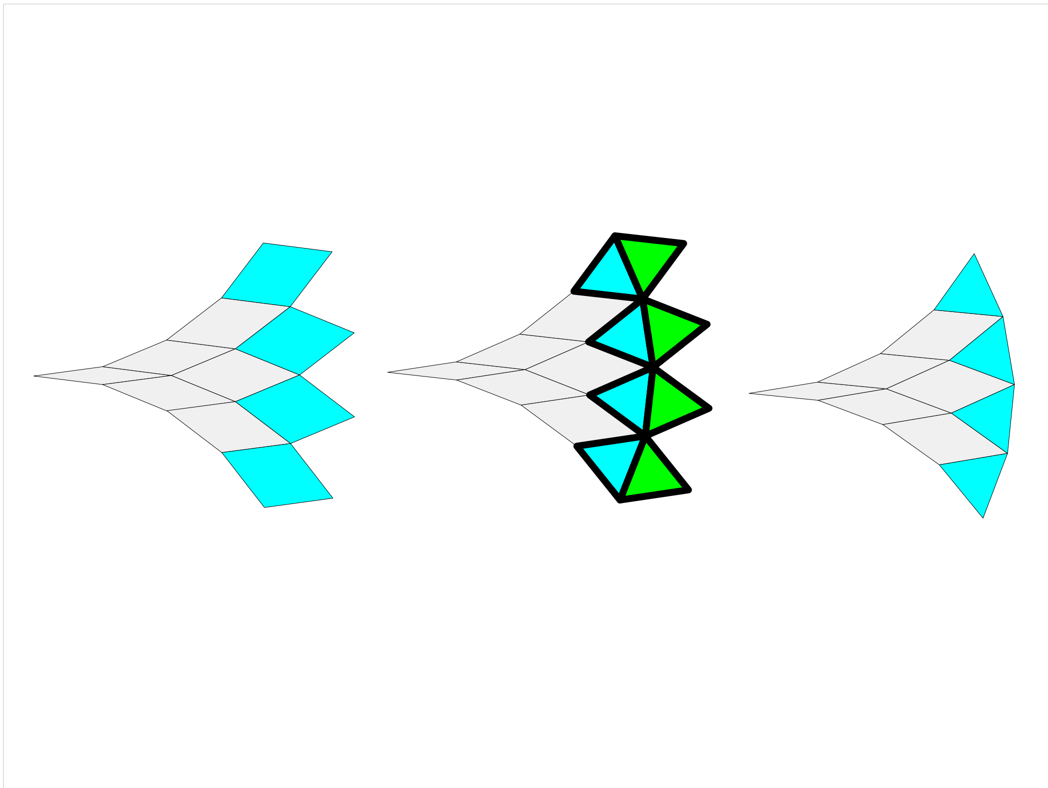
Take a rhom with one internal angle pi/m.

Add two rhombs with internal angles 2pi/m.

Add three rhombs with internal angles 3pi/m.

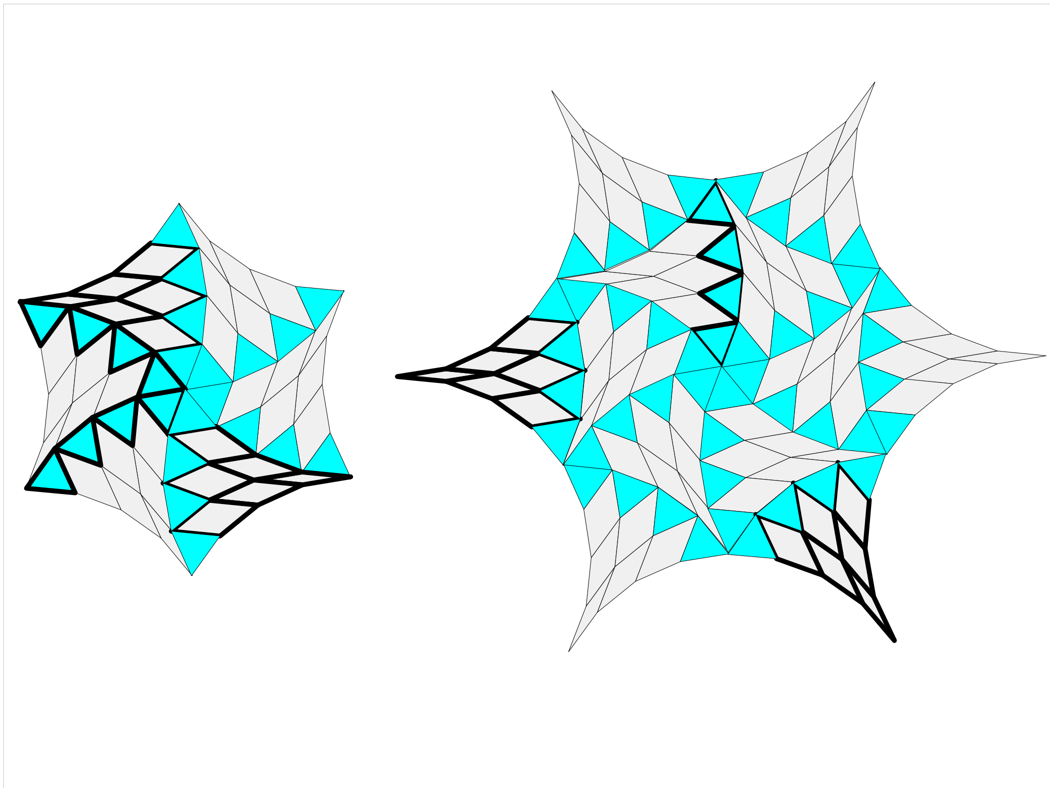
Repeat until we add m-1 rhombs with angles (m-1)pi/m.

You get this kind of figure.

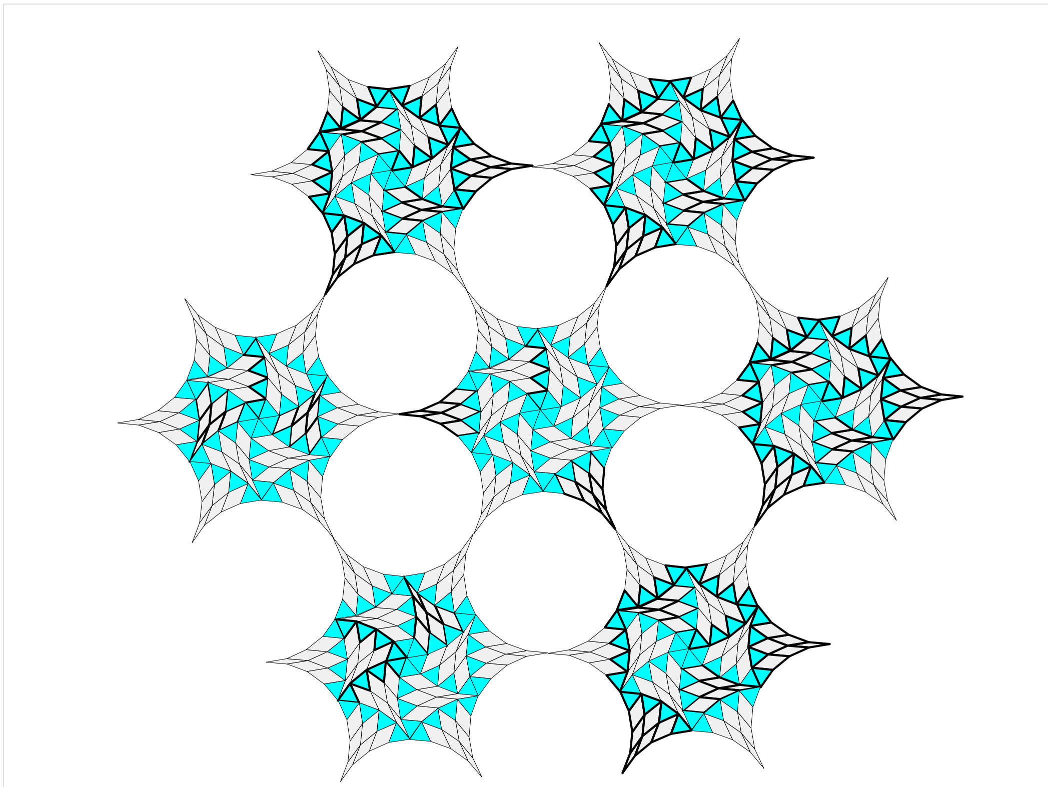


If we are constructing a rhombic fan or order m, and m is a multiple of 3, then a third of the way through the construction process we will encounter a set of rhombs with internal angles pi/3. This is a special kind of rhomb because it can be split into 2 equilateral triangles.

If we do this and stope the fan construction process we get a rhombic trangle of order 3n.



Rhombic triangles have many interesting properties. You can combine 6 of them to form a rhombic hexagon and add 6 more to form a rhombic star.



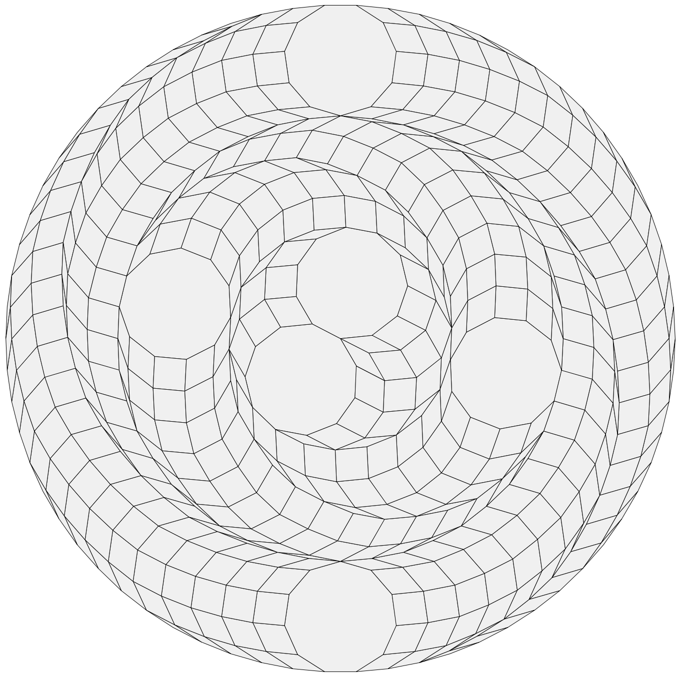
A rhombic star of order 3n almost tiles the plane. It just needs the help of a regular polygon of order 6n.

So any regular polygon of order 6n can be extended to a periodic tiling.

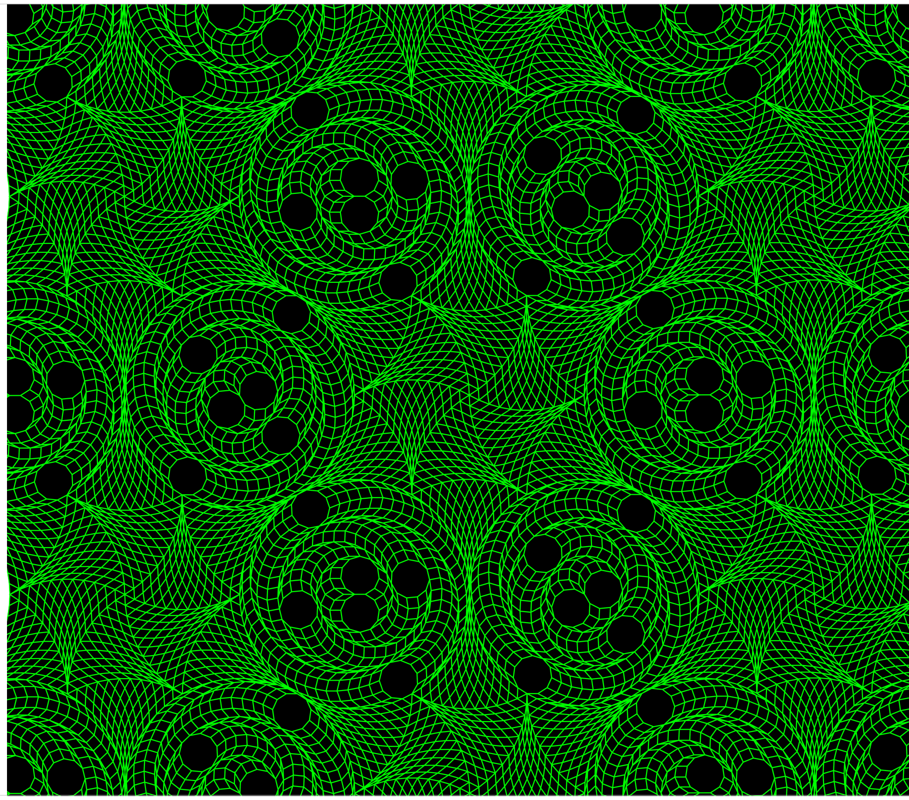
Decomposition algorithm

A regular polygon of composite order mn (m > 1, n > 2) can be decomposed into m n-gons and rational rhombs.

But we can say much more because of the decomposition algorithm I present on my site.

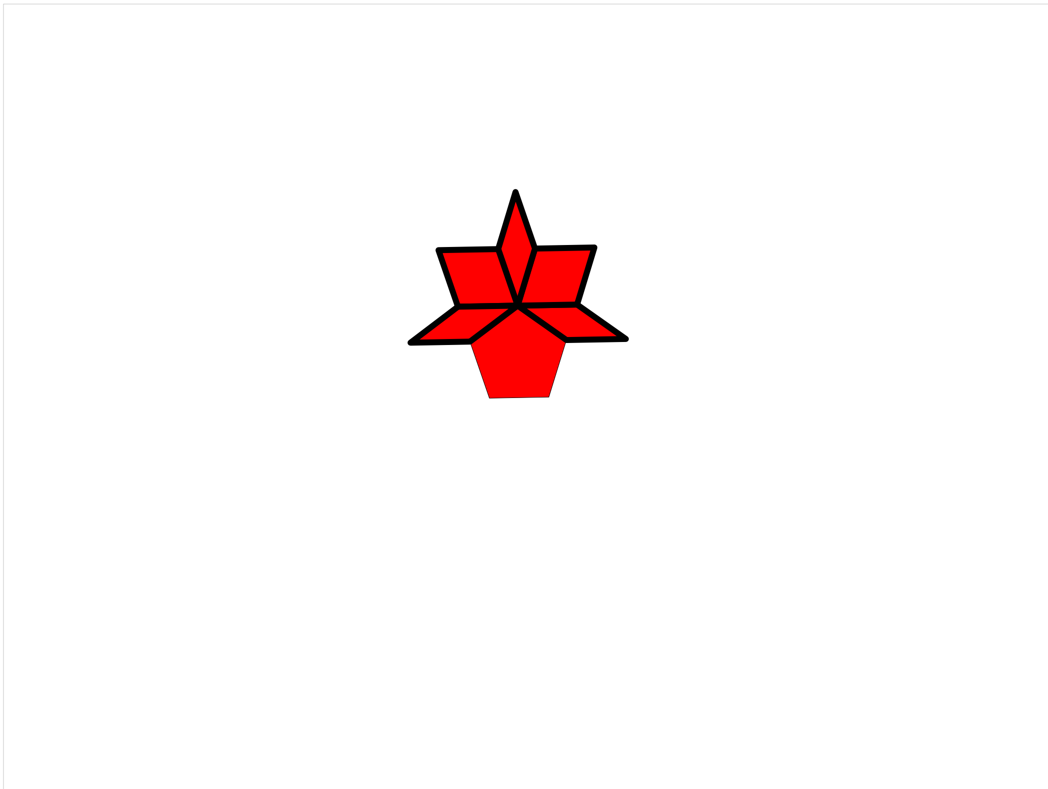


Here is the 66-gon decomposed into 6 hendecagons and rhombs.



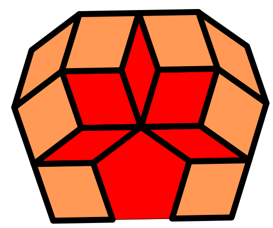
The rhombic star of rder 33 tiles the plane with the regular polygon of order 66. Here I've decomposed each 66-gon into hendecagons.

By a similar process any regular polygon can be extended to a periodic tiling.

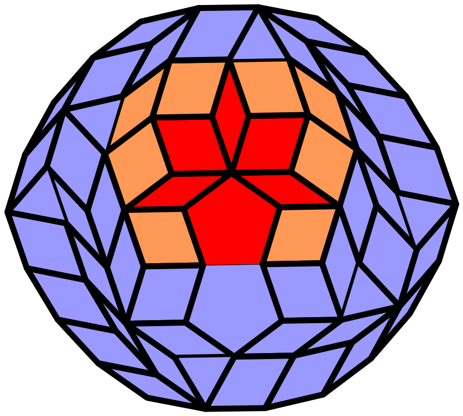


So now we need an algorithm to extend any vertex figure of regolar polygons and rational rhombs to a regular polygon.

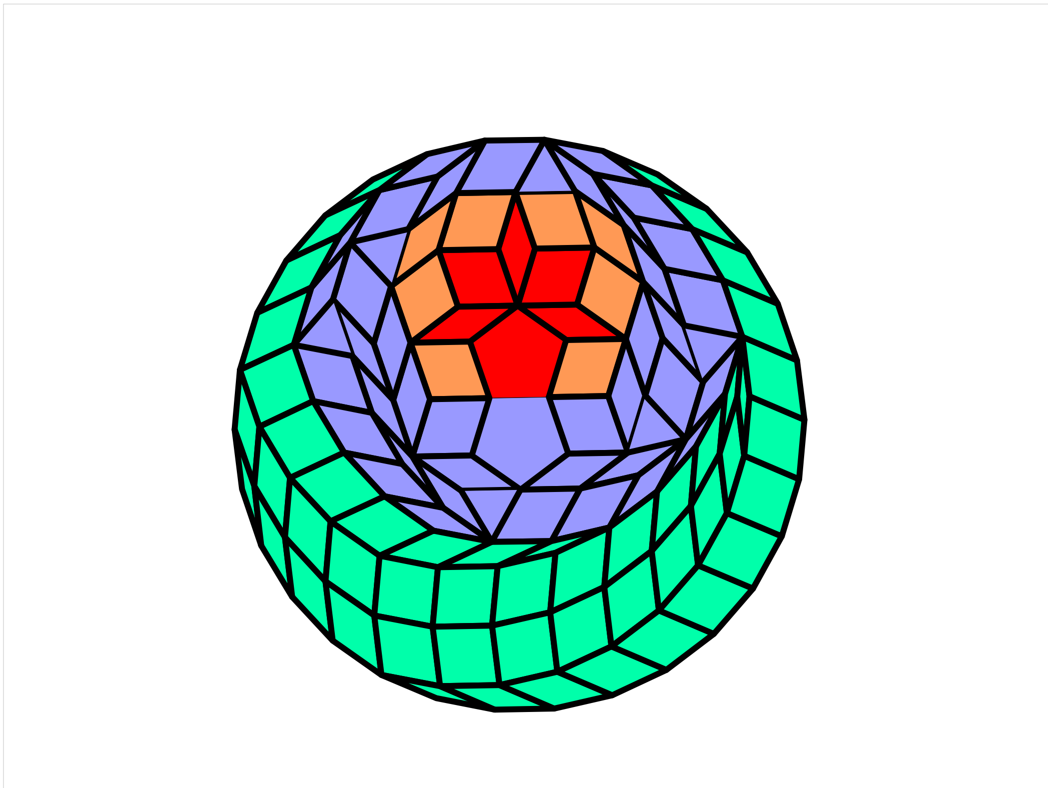
Here's an example.



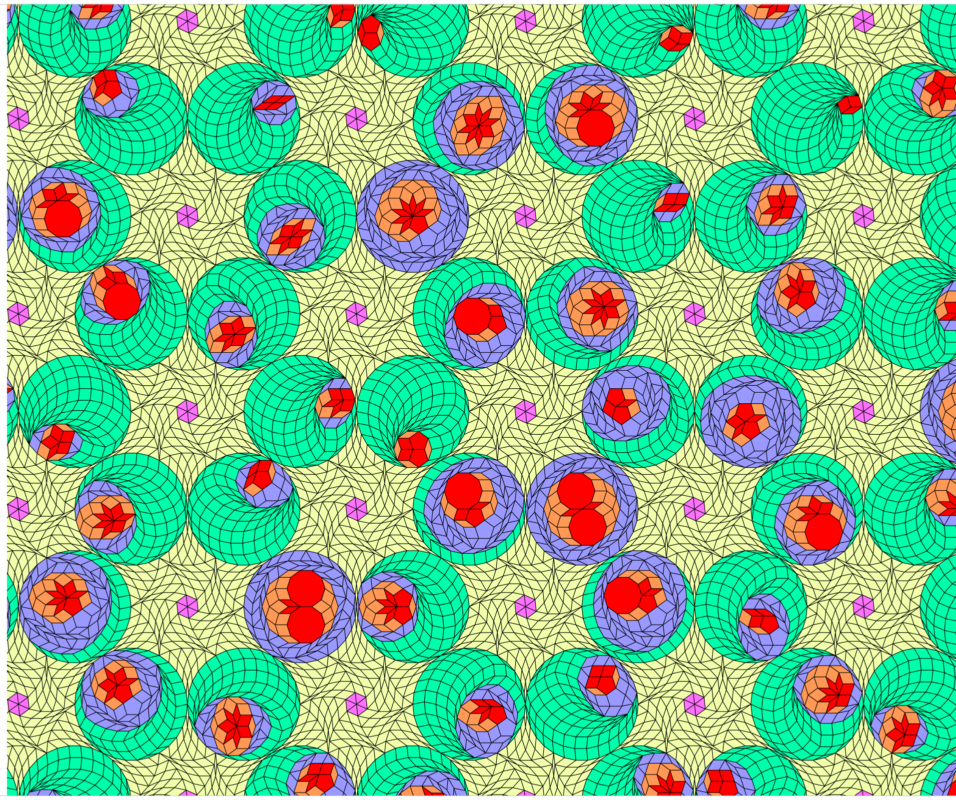
Convexification.



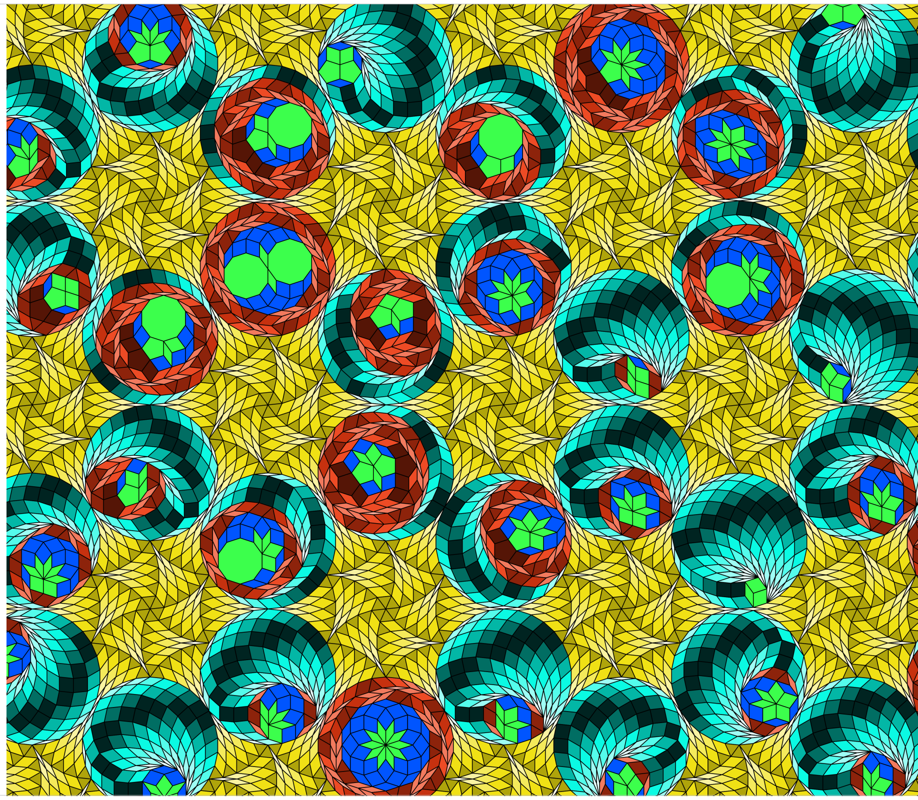
Zonagonification – usually requires regular polygons.
In this case a pentagon and triangles are added.



Regularisation. The vertex figure has been extended to a 30 gon.



There are 121 vertex figures with regular polygons and rational rhombs with internal angle a mulriple of pi/5. Here are some extended to a tiling.



This recoloured version is in the Bridges 2013 art show.