Imperfect Congruence: Tiling with Regular Polygons and Rhombs

Kevin Jardine
Bloemistenlaan 30 • Leiden 2313 BB • Netherlands
kevin@radagast.biz

Abstract

Edge-to-edge plane tilings of regular polygons can include only squares, triangles, hexagons, octagons, and dodecagons. The possibilities become far larger, striking and beautiful if we add rhombs to the prototile set.

The mathematician and astronomer Johannes Kepler was the first person to publish a detailed study of regular polygon tilings of the plane and sphere in the second chapter of *Harmonices Mundi* (The Harmony of the World) in 1619 [1]. Kepler's main purpose was to present a catalog of what he called *perfect congruences.* A perfect congruence is an arrangement of regular polygons around a single vertex that can be extended to a periodic tiling with the same arrangement at all vertices. It turns out there are 11 such arrangements in the plane. Today the tilings that result from these arrangements are called the *uniform tilings*.

Kepler paid less attention to what he called *imperfect congruences*. An imperfect congruence is an arrangement of polygons around a vertex that cannot be extended to a uniform tiling. Let us define a vertex figure as an arrangement of polygons around a single vertex such that each polygon has exactly two edges adjacent to other polygons and intersecting in the central vertex. Research published in the early twentieth century shows that there are 21 possible vertex figures of regular polygons in the plane [2]. Eleven of these correspond to Kepler's perfect congruences and 10 are imperfect congruences.

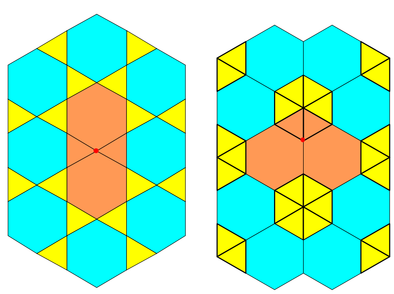

Figure 1 :*The perfect congruence 3.6.3.6 (left) versus the imperfect 3.3.6.6 (right).*

Imperfect Congruences

Vertex figures of regular polygons are conventionally represented by a *vertex type* describing a *bracelet* of integers. A bracelet is a structure which is invariant under cyclic permutation or reflection. Each integer in the bracelet represents the order of a regular polygon. The bracelet can be written as a sequence of integers separated by dots. For example, the vertex type 3.6.3.6 represents one of Kepler's perfect congruences. Because the vertex type is a bracelet, 3.6.3.6 is identical to any cyclic permutation or reflection, for example, 6.3.6.3. However, as is shown in figure 1, it is not identical to 3.3.6.6 which is an imperfect congruence and can only be extended to a full tiling using vertices of other types (in this case, 3.3.3.3.3.3).

Of the ten imperfect congruences, four can be extended to periodic tilings using one or more additional vertex types. Six are *forbidden* vertex figures – they cannot appear in any edge-to-edge plane tiling of regular polygons. These are 4.5.20, 5.5.10, 3.7.42, 3.8.24, 3.9.18 and 3.10.15.

All the non-forbidden vertex figures consist of triangles, squares, hexagons, octagons and dodecagons.

Adding Rational Rhombs to the Prototile Set

The variety of patterns possible in regular polygon tilings is quite limited. We can greatly extend the possibilities by adding rational rhombs to the prototile set. A rhomb is a generalisation of the square. Like a square, a rhomb has four equal sides. Unlike a square, a rhomb has two distinct internal angles. If one of these angles is α, the other must be π-α.

A *rational rhomb* is a rhomb whose internal angles are rational multiples of π. As explained in detail on the *Imperfect Congruence* website, each of the six forbidden vertex figures can be extended to periodic tilings of the plane using a small set of rational rhomb prototiles [3]. This result can be proved easily by exhibiting translational units for tilings containing each of the forbidden vertex figures. These units can be repeated indefinitely to form the tiling. Here are such units:

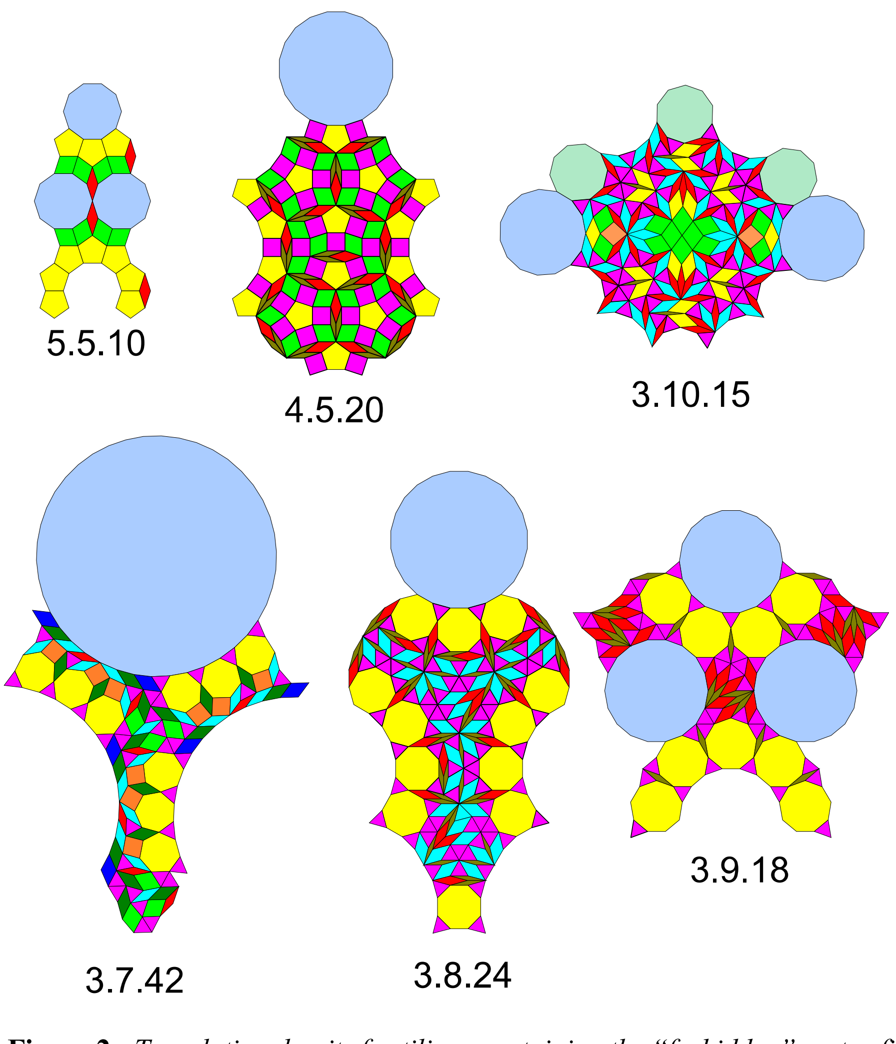

Figure 2 : *Translational units for tilings containing the "forbidden" vertex figures*

Mixed Vertex Figures

Now that we have allowed rational rhombs into the prototile sets, can any vertex figure consisting of rational rhombs or regular polygons be extended to a periodic tiling of the plane? I present an algorithm on the *Imperfect Congruence* website that appears to be able to construct a periodic tiling given any such vertex figure. There is no room to describe the algorithm in this paper – interested people are invited to visit the website.

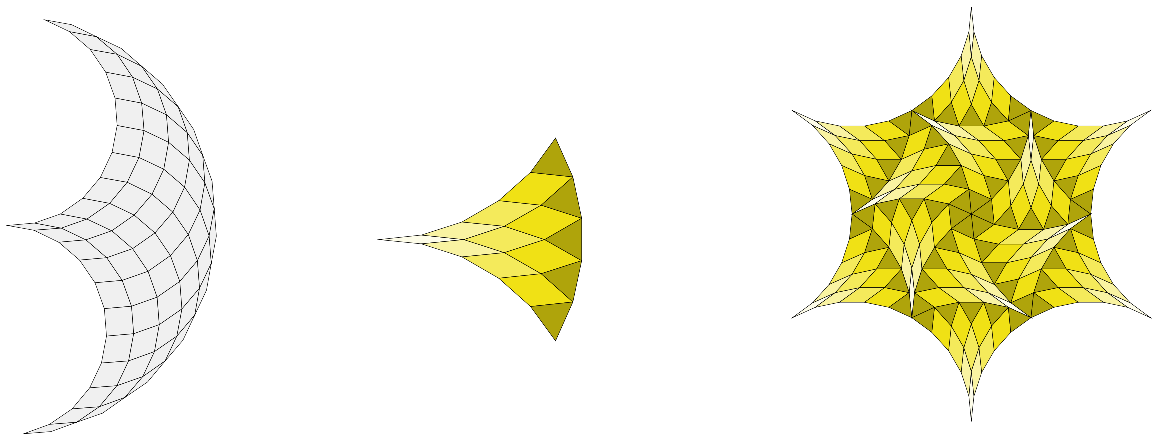

Figure 3 : *Rhombic fan, triangle and star*

As a small taste, consider the following: given an integer m >2, take the rhomb with angle π/m. Then add two rhombs with angle 2π/m, three rhombs with angle 3π/m and so on until you have added m-1 rhombs with angle (m-1)π/m. This structure fans out from the initial π/m rhomb so I call it a *rhombic fan of order m*. (See Figure 3.)

One important variation works only for fans of order m = 3n. In this case we start with the rhomb with small angle π/m but stop a third of the way through the process, when we reach the n rhombs of angle nπ/m. Since m = 3n, these rhombs always have the angle nπ/3n = π/3. This rhomb has the special property that it consists of two triangles joined together. Put another way. the equilateral triangle is half of the rhomb with small angle π/3.

Now split the π/3 rhombs into two triangles and then drop the second triangle from each pair. This creates a tile patch of rhombs and triangles that I call a *rhombic triangle*. You can combine 12 rhombic triangles to form a *rhombic star*.

As shown in *Imperfect Congruence*, a rhombic star of order 3n can tile the plane with a regular polygon of order 6n and any regular polygon of composite order mn (n > 2) can be decomposed into rhombs and m n-gons using a variation of the decomposition algorithms published by Sampath Kannan and Danny Soroker [4] and Richard Kenyon [5].

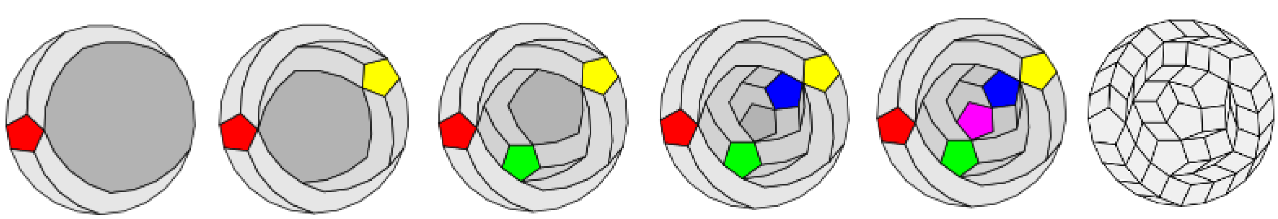

Figure 4 : *A 25-gon is split into five pentagons and rhombs*

These results allow the construction of many interesting and beautiful tilings with any regular polygon, including this one with hendecagons (11-sided polygons). This tiling consists of a rhombic star of order 33 tiled with a regular polygon of order 66. Each 66-gon has been decomposed into rhombs and 6 hendecagons.

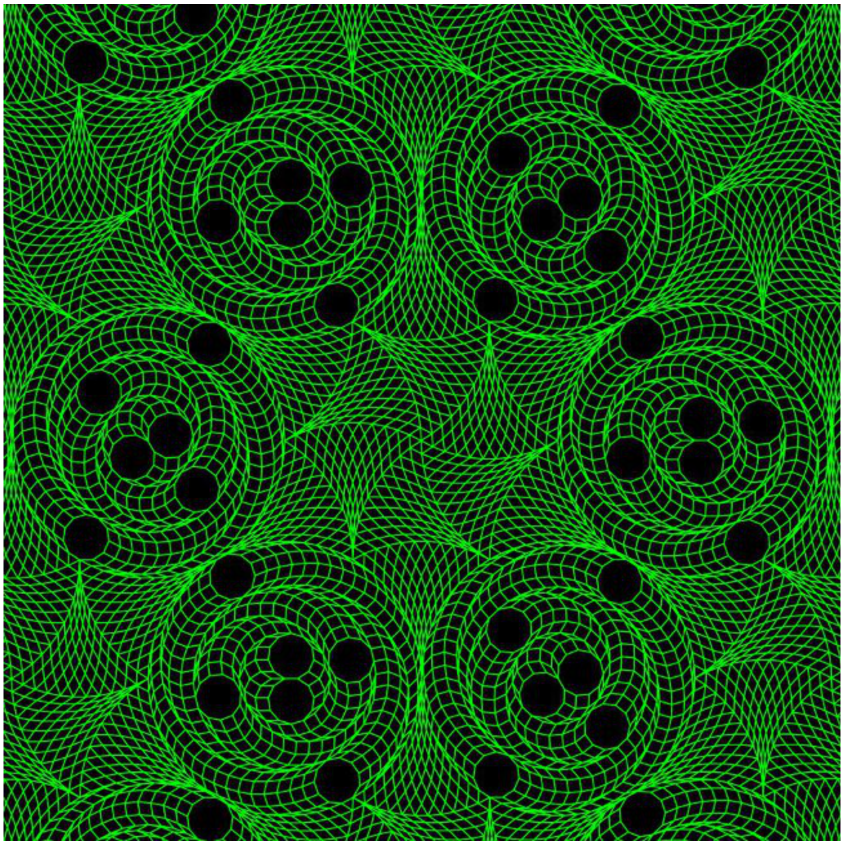

Figure 5 : *A periodic tiling incorporating 11-sided polygons*

Thank you for the feedback from two anonymous reviewers on an earlier version of this paper and the encouragement of Craig S. Kaplan from the University of Waterloo.

References

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[4] Sampath Kannan and Danny Soroker. "Tiling polygons with parallelograms", *Discrete & Computational Geometry*, Volume 7, Issue 1, pp 175-188, 1992.

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